STUDIES ON SIGNAL PROCESSING OF CYCLOSTATIONARY PROCESSES

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Abstract

Among non-stationary processes, there exist cyclostationary processes whose means and auto-correlation functions are periodic in time. These cyclostationary processes are appropriate to model stochastic processes with periodic variations. So these are important in the fields of signal processing and stochastic signal analysis because there are many periodic signal processing operations. The main purpose of this thesis is to study the theory of cyclostationary processes and to develop its applications to time delay estimation, optimization of filter banks, and backward periodic AR processes.

The cyclostationary process with zero mean is characterized by the cyclic auto-correlation in the time domain or by the spectral correlation density in the transform domain. In Chapter 2, the sampling theorem of cyclostationary processes are presented. This shows the spectral relation between the original continuous-time cyclostationary process and the sampled discrete-time cyclostationary process. A new derivation of Gladyshev's relation is also presented. This shows the spectral relation between discrete-time cyclostationary processes and discrete-time multichannel stationary processes. As one direct application of cyclostationary processes, a problem of estimating the time difference of arrival (TDOA) of a communication signal between two sensors is considered. This problem has been studied in radar or sonar signal processing because the direction of arrival of the wavefront can be estimated from this TDOA. By using the sampling theorem and the cyclostationary analysis, the maximum likelihood estimator is derived. The performance of this method is better than the existing ones under certain conditions. This is clarified by the results of computer simulations.

Recently, in signal processing, there are a great deal of works concerning filter banks in multirate signal processing. Most of them have been studied from the deterministic point of view, or if from the stochastic point of view, the cyclostationary spectral analysis is not used explicitly. But the output of a filter bank for a stationary input is generally cyclostationary. So, in Chapter 3, the general spectral relation between the input signal and the output signal is derived by using the cyclostation-

ary spectral analysis. From this analysis, it is shown that the output of an alias free filter bank for any stationary input is stationary. Then the perfect reconstruction (PR) condition is restated from the stochastic point of view. Next, we derive the averaged mean squared reconstruction error when the high pass band signal in a two-band filter bank is dropped. This is used as a criterion to optimize two-band filter banks under the PR condition. From this criterion, the optimal biorthogonal filter banks are obtained. By adding additional constraints to the filter coefficients, the criterion of conjugate quadrature filter banks and that of perfect reconstruction linear phase filter banks are respectively obtained. Then the obtained PR filter banks are compared in terms of other criteria.

To generate or to analyze cyclostationary processes parametrically, periodic autoregressive moving average (ARMA) processes are often used. At first, in Chapter 4, these are studied by using the theory of multirate systems. As special cases of ARMA processes, periodic AR processes are important in theory and in practice. These have some statistical properties like those of conventional AR processes. For example, there exists the circular Levinson algorithm to obtain the coefficients of the transfer function from the covariances efficiently like the Levinson-Whittle-Wiggins-Robinson algorithm for conventional AR processes. It is known that there exist backward AR processes corresponding to AR processes. Similar to this result, it is shown that backward periodic AR processes can be constructed from the auxiliary coefficients used in the circular Levinson algorithm. Then the orders of the backward periodic AR process are shown to be different from those of the corresponding periodic AR process. A numerical example and statistical properties of the estimated coefficients from a sample of finite size are also presented.

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Contents

Chapter 1

INTRODUCTION

In the fields of signal processing and signal analysis, when the interested signal is considered to be a stochastic signal, it is often modeled as a stationary process. It is one of the second-order processes whose mean is constant and whose auto-correlation function is only dependent on the time difference of two time variables. The stationary process with zero mean is characterized by the auto-correlation in the time domain or by the spectral density in the transform domain. This simple modeling has achieved many useful applications in spectral analysis [19] [32]. But, in practice, most signals are not stationary. So, it has been often discussed the importance of these non-stationary processes.

Among non-stationary second-order processes, there exists a cyclostationary process whose mean is periodic in the time variable and whose auto-correlation function is periodic in two time variables. This process is appropriate to model stochastic processes with periodic variations. So, this process is important in the fields of signal processing and stochastic signal analysis because there are many periodic signal processing operations like sampling, modulation, and coding or the physical periodic phenomenon like the monthly data of the temperature. Furthermore, as described later, when the signal is considered to be a stochastic signal, the cyclostationarity plays a great role in multirate systems and linear periodically time-varying systems, which are ones of the current main studies in signal processing and stochastic signal analysis. Thus the importance of cyclostationary processes may be growing in the future. Therefore it is necessary to know and use this cyclostationarity as a basic tool of signal processing and stochastic signal analysis.

The main purpose of this thesis is to study and develop the theory of cyclostationary processes and to derive new applications and analysis. As one direct application of cyclostationary processes, estimation of the time delay of a cyclostationary signal is considered. Next, by using the cyclostationary spectral analysis, optimization of perfect reconstruction filter banks is studied. Lastly, as one parametric analysis for cyclostationary processes, new results about periodic autoregressive processes are derived. In the following, the introduction of each topic is discussed.

1.1 Cyclostationary Processes

The cyclostationary process can be interpreted as an extension of the stationary process. The cyclostationary process with zero mean is characterized by the cyclic auto-correlation in the time domain, which is a generalization of the auto-correlation, or by the spectral correlation density in the transform domain, which is a generalization of the spectral density.

The spectral correlation density for discrete-time cyclostationary processes was presented by Glasyshev [17] (1961). It was shown there is a one-to-one correspondence between a scalar cyclostationary process and a multichannel stationary process. It was also shown the relation between the spectral correlation density of a scalar cyclostationary process and the spectral density matrix of the corresponding multichannel stationary process. This is called Gladyshev's relation. Gladyshev's relation is useful because the existing theory and analysis of multichannel stationary processes can be used after transforming the scalar cyclostationary process into the multichannel stationary process, and vice versa. On the other hand, the spectral correlation density for continuous-time cyclostationary processes was presented by Ogura [27] (1971) and by Gardner and Franks [10] (1975), respectively. These spectral representations enable us new spectral analysis for cyclostationary processes.

From the middle of the 1970's, it was noticed that there are many cyclostationary signals treated in signal processing. In the 1980's, the cyclostationary modeling has begun to be used positively. Gardner and others added the theoretical bases of continuous-time cyclostationary processes [11] [12] [13]. For discrete-time cyclostationary processes, Sakai [35] (1991) restated Gladyshev's relation as a form easily to be used.

The applications using this cyclostationarity have been reported in many literature. In this thesis, at first, a problem of estimating the time difference of arrival (TDOA) of an interested communication signal with additive noises between two sensors is considered. This problem has been studied in radar or sonar signal processing because the direction of arrival of the wavefront can be estimated from this TDOA under certain conditions. Using the stationary modeling, many methods have been already derived [4]. But, in fact, most communication signals exhibit the cyclostationarity because of some periodic operations. A new method using this cyclostationarity was derived by Gardner and Chen [14] (1988), which is reported to be highly tolerance to severely corruptive noise and interference. Another method was reported by Xu and Kailath [44] (1990). But they only use a part of the information of the received signals, that is, one slice of the spectral correlation density. In this thesis, expecting that better estimates can be obtained if all of the information of the received signals is used, we derive the maximum likelihood estimator that uses all of the spectral correlation density. This is a generalization of the one obtained by Wax [45] for stationary signals. The performance of this method is better than the previous ones particularly when the signal/noise ratio (SNR) is low. This is clarified by the results of the computer simulations.

1.2 Filter Banks

Recently, in signal processing, there are a great deal of works concerning the multirate subband filtering method [2] [39] [40]. In this method, in the analysis part, after being band pass filtered a signal is first downsampled by decimator and then divided into subband signals. After quantization, these signals are transmitted to the synthesis part. In the synthesis part, these signals are first upsampled by interpolator. Then they are filtered by band pass filters and finally added. This system is called the filter bank.

The two-band quadrature mirror filter (QMF) banks introduced in the middle of the 1970's were originally used for the subband coding by Crochiere and others [5] (1976). This subband coding has many advantages over waveform coding. For example, the reconstruction error due to the quantizing can be reduced. It can be also used to compress a signal into some subband signals with little loss of information. Crochiere and Rabiner also wrote the first book in this field [6] (1983), which perhaps stimulated many researchers.

The special filter bank whose output is time delayed version of the input is called the perfect reconstruction (PR) filter bank, which plays a great role in multirate systems. Since it was shown that the QMF banks can not have the PR property, the conjugate quadrature filter (CQF) banks were derived and have been used for PR filter banks.

After the success of the subband coding, there have been many attempts of using filter banks in other fields of signal processing. With the development of micro electronics, adaptive signal processing has begun to be used. Because it is often the case that a large number of filter coefficients has to be used when there is the limit of the hardwares, it is difficult to implement some methods. So, PR filter banks are used to reduce the filter coefficients per band [20] (1991) [16] (1992) since the number of the coefficients of subband signals of M-band filter banks is reduced by the factor M of those of the original signal.

As a new tool of signal analysis, by the end of the 1980's, the wavelet analysis has received much attention because it has the potential of the local time/frequency analysis. It was shown that the theory of PR filter banks is closely related to those of multiresolution analysis and wavelet analysis, which had been studied independently of filter banks. The main contribution of these are by Mallat [23] [24] (1989) and by Daubechies [8] (1988) [9] (1992), respectively. Daubechies [8] showed that wavelet mother functions can be obtained from two-band CQF banks by adding additional conditions.

As seen above, PR filter banks are used in many fields of signal processing. The theory of filter banks has been usually derived under the assumption that the input is a deterministic signal. For two-band CQF banks, the basic result was proved independently by Smith and Barnwell [37] (1984) and by Mintzer [26] (1985). This result was extended to M-band CQF banks and generated many applications. Recently since filters of two-band CQF banks can not have linear phase except some trivial cases, the PR filter banks without the CQF condition have been also studied by Vetterli and Herley [43] (1992), which are called the biorthogonal filter banks.

In signal processing, since there are some freedoms of coefficients of PR filter banks, the "good" PR filter bank can be chosen according to the object of using the PR filter bank. To do so, the stochastic input signals are often used to derive a certain criterion.

For two-band CQF banks, in order to minimize the effect of the quantizing, or equivalently to minimize the reconstruction error when the high pass band signal is dropped, the optimal coding gain two-band CQF banks were constructed independently by Akansu and others [3] (1992) and by Vandendorpe [41] (1992). There only the stationary spectral analysis with the CQF property was used. But, if the input is a stationary stochastic signal, then the output of a filter bank is no longer stationary. Actually the output signal is cyclostationary with period M where M is the rate of decimation and interpolation. To fully characterize the output, it is necessary to know the spectral correlation density matrix of this cyclostationary output signal. Thus we derive this matrix by using the cyclostationary spectral analysis. Then from this analysis, filter banks can be analyzed in terms of the stochastic point of view. As one application of this analysis, we derive the averaged mean squared reconstruction error when some subband signals are dropped. This error is used as a criterion to optimize a two-band filter bank under the PR condition when the high pass band signal is dropped. From this criterion, the optimal biorthogonal filter banks, the optimal CQF banks and the optimal PR linear phase filter banks are constructed respectively.

1.3 Parametric Analysis of Cyclostationary Processes

Linear periodically time-varying (LPTV) systems whose coefficients of the transfer function are periodic in time are also closely related to both cyclostationary processes and multirate systems. In fact, the output of a LPTV system with period M is cyclostationary with the same period M; the LPTV system with period M can be expressed as a special case of M-band filter banks.

Among many classes of the LPTV systems, there exist periodic autoregressive moving average (ARMA) systems with the order (p_t, q_t) where p_t and q_t are periodic in time. The output $y(t)$ of a ARMA system, called the periodic ARMA process, at the time t is linearly dependent on the output $y(t-1), \ldots, y(t-p_t)$ and the input $\epsilon(t), \ldots, \epsilon(t-q_t)$. Because of the simplicity of mathematical analysis, ARMA processes are used to produce certain cyclostationary processes and to analyze them parametrically. Vecchia [42] (1985) presented the maximum likelihood estimate for periodic ARMA processes and used this method to analyze a seasonal stream flow series of a river. Sakai [35] (1991) showed the spectral density matrix of a periodic ARMA process.

As special cases of ARMA processes, the periodic AR processes that have only one moving average term are important in theory and in practice. They were originally studied by Pagano [31] (1978). Corresponding to the Levinson-Whittle-Wiggins-Robinson (LWR) algorithm [47] for conventional AR processes, the circular Levinson algorithm was derived by Sakai [33] (1982) to obtain the coefficients of the transfer function from the covariances efficiently. Also by Sakai [34] (1983) [36] (1991), the circular Levinson algorithm was used to study further properties of periodic AR processes. There it was shown that periodic AR processes have some statistical properties like those of conventional AR processes.

It is known that there exist backward AR processes corresponding to conventional AR processes, which are used for lattice filters and so on [19]. In this thesis, by using the circular Levinson algorithm, we show a new fact about the existence and the construction of the backward periodic AR processes. Then the orders of the backward periodic AR process are shown to be different from those of the corresponding periodic AR process. We also derive statistical properties of the estimates of their coefficients from a sample of finite size.

1.4 Paper Outline

This thesis organized as described in the following.

Chapter 2 is directly concerned with cyclostationary processes. After the concept and properties of continuous-time cyclostationary processes are reviewed, some important examples of continuous-time cyclostationary processes are shown. Next, discrete-time cyclostationary processes are also reviewed and Gladyshev's relation is stated with different derivation from that of Sakai in [35]. Then the sampling theorem of cyclostationary processes is derived. The estimates of the statistics of cyclostationary processes are briefly discussed. By using the above results, the maximum likelihood estimator of the time difference of arrival of an interested communication signal between two sensors is derived. The results of the computer simulations are also presented.

Chapter 3 studies filter banks in multirate systems. Some important results of multirate systems are introduced. Using the cyclostationary spectral analysis, the spectral correlation density matrix of the output of filter banks is derived. Then it is proved that the output of an alias free filter bank for any stationary input is stationary. The PR condition is restated from the stochastic point of view. Next, we derive the averaged mean squared reconstruction error when some subband signals are dropped. After the PR filter banks are discussed, this error is used as a criterion to optimize a two-band filter bank under the PR condition when the high pass band signal is dropped. Other criteria are also shown, from which the obtained PR filter banks are compared. Some numerical results are also presented for the optimal biorthogonal filter banks and the optimal PR linear phase filter banks.

Chapter 4 deals with LPTV systems for parametric analysis of cyclostationary processes. The periodic ARMA systems are briefly reviewed and their connection to cyclostationary processes and multirate systems are discussed. Then the spectral correlation density matrix of the output of a periodic ARMA system is derived. Next, for periodic AR processes, after their relation to multichannel AR processes is mentioned, the existence of backward periodic AR processes is shown. Then after the circular Levinson algorithm is introduced, backward periodic AR processes are constructed from the auxiliary coefficients used in the circular Levinson algorithm. A numerical example and statistical properties of estimates of coefficients are also presented.

Chapter 5 is the conclusion of this thesis.

Chapter 2

CYCLOSTATIONARY PROCESSES

This chapter is concerned with cyclostationary processes. In Section 2.1, the theory of continuous-time cyclostationary processes and discrete-time cyclostationary processes are introduced, respectively. Some examples of continuous-time cyclostationary processes are shown. Then the sampling theorem of cyclostationary processes is derived. The estimates of the statistics of cyclostationary processes are also discussed. In Section 2.2, as one application of cyclostationary processes, the estimation of the time difference of arrival of a cyclostationary communication signal between two sensors are studied. Then we derive the maximum likelihood estimate and evaluate this by computer simulations.

2.1 Theory of Cyclostationary Processes

2.1.1 Continuous-time cyclostationary processes

Now we consider continuous-time cyclostationary processes.

Define the mean and the auto-correlation of a process $x(t)$ as

$$E[\, x(t) \,] = m(t) \tag{2.1}$$

$$E[\, x(t)x^*(s) \,] = R(t, s) \tag{2.2}$$

where $E[\cdot]$ and the asterisk denote the expectation operator and the complex conjugate, respectively.

A continuous process $x(t)$ is said to be cyclostationary with period T if its mean and auto-correlation are periodic with period T, that is,

$$m(t+T) = m(t) \tag{2.3}$$

$$R(t+T, s+T) = R(t,s) \tag{2.4}$$

for all t and s. For convenience, we assume $m(t) = 0$.

By changing the variables, (2.4) can be rewritten as $R(t+\tau/2, t-\tau/2)$. For fixed τ, $R(t+\tau/2, t-\tau/2)$ is periodic in t with period T. Under the assumption that its Fourier series representation for this periodic function converges, $R(t+\tau/2, t-\tau/2)$ can be expressed by

$$R(t+\frac{\tau}{2}, t-\frac{\tau}{2}) = \sum_{\alpha} R^{\alpha}(\tau) e^{j2\pi\alpha t} \tag{2.5}$$

where α ranges over all integer multiples of $1/T$. This $R^{\alpha}(\tau)$ is called the cyclic auto-correlation function at the cyclic frequency α [15]. Conversely, the cyclic auto-correlation $R^{\alpha}(\tau)$ is expressed as

$$R^{\alpha}(\tau) = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} R(t+\frac{\tau}{2}, t-\frac{\tau}{2}) e^{-j2\pi\alpha t} dt \quad \alpha = \frac{n}{T} \ (n : integer). \tag{2.6}$$

Assume that the Fourier transformation of the cyclic auto-correlation;

$$S^{\alpha}(f) = \int_{-\infty}^{\infty} R^{\alpha}(\tau) e^{-j2\pi f\tau} d\tau \tag{2.7}$$

exists. This $S^{\alpha}(f)$ is called the spectral correlation density at the cyclic frequency α [15]. It should be noted that conversely $R^{\alpha}(\tau)$ is rewritten as

$$R^{\alpha}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S^{\alpha}(f) e^{j2\pi f\tau} df. \tag{2.8}$$

Now let $x(t)$ be stationary with zero mean. The auto-correlation of this stationary process, denoted by $R_x(\tau)$, satisfies

$$R(t,s) = R_x(t-s). \tag{2.9}$$

By substituting this into (2.6), it can be easily shown that

$$R^{\alpha}(\tau) = \begin{cases} R_x(\tau) & \alpha = 0 \\ 0 & \alpha \neq 0 \end{cases}, \tag{2.10}$$

then, from (2.7),

$$S^{\alpha}(f) = \begin{cases} S(f) & \alpha = 0 \\ 0 & \alpha \neq 0 \end{cases} \tag{2.11}$$

where $S(f)$ denotes the spectral density of the stationary process $x(t)$.

As seen above, if $x(t)$ is a stationary stochastic signal, then the cyclic auto-correlation $R^{\alpha}(\tau)$ and the spectral density $S^{\alpha}(f)$ at $\alpha = 0$ reduce to the conventional auto-correlation and the conventional spectral density, respectively. Therefore it can be said that the cyclic auto-correlation and the spectral correlation density include the auto-correlation and the spectral density as a special case. So, the cyclostationary process can be interpreted as a generalization of the stationary process.

Similarly to (2.6) and (2.7), the cyclic cross-correlation and the spectral cross-correlation density are defined by

$$\begin{aligned} R^{\alpha}_{yx}(\tau) &= \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} E[\, y(t+\frac{\tau}{2})x^{*}(t-\frac{\tau}{2})\,]e^{-j2\pi\alpha t}dt \\ S^{\alpha}_{yx}(f) &= \int_{-\infty}^{\infty} R^{\alpha}_{yx}(\tau)e^{-j2\pi f\tau}d\tau, \end{aligned}$$

respectively.

If the spectral correlation cross-correlation between $x(t)$ and $y(t)$ is identically zero at a certain α, $x(t)$ and $y(t)$ are said to be mutually cyclic incoherent at the cyclic frequency α [15]. It should be noted that if $x(t)$ and $y(t)$ are incoherent then the cyclic cross-correlation and the spectral cross-correlation density are identically zero for all α.

2.1.2 Examples of continuous-time cyclostationary processes

We show some examples of continuous-time cyclostationary communication signals that are important in communication signal processing.

We assume that $a(t)$ is a real continuous-time stationary process with zero mean, the auto-correlation $R_a(\tau)$, and the spectral density $S_a(f)$ with frequency band $[-1/2T, 1/2T]$ where T denotes the period of the interested cyclostationary signal.

Consider the following amplitude modulated (AM) signal with carrier frequency f_0;

$$x(t) = a(t)\cos(2\pi f_0 t + \phi_0). \tag{2.12}$$

From

$$\begin{aligned} m_x(t) &= E[\,a(t)\,]\cos(2\pi f_0 t + \phi_0) = 0 \\ R_x(t+\frac{\tau}{2}, t-\frac{\tau}{2}) &= \frac{1}{2}R_a(\tau)\{\cos(4\pi f_0 t + 2\phi_0) + \cos 2\pi f_0 \tau\}, \end{aligned}$$

$R_x(t+\tau/2, t-\tau/2)$ is a periodic function in t with period $T = 1/2f_0$. Then, from the definition, the cyclic auto-correlation is given by

$$R_x^\alpha(\tau) = \begin{cases} \frac{1}{2}R_a(\tau)\cos 2\pi f_0\tau & \alpha = 0 \\ \frac{1}{4}R_a(\tau)e^{\pm j\phi_0} & \alpha = \pm 2f_0 \\ 0 & \text{otherwise} \end{cases},$$

and the spectral correlation density is given by

$$S_x^\alpha(f) = \begin{cases} \frac{1}{4}(S_a(f+f_0) + S_a(f-f_0)) & \alpha = 0 \\ \frac{1}{4}S_a(f)e^{\pm j\phi_0} & \alpha = \pm 2f_0 \\ 0 & \text{otherwise} \end{cases}. \tag{2.13}$$

Next, consider the pulse amplitude modulated (PAM) signal;

$$x(t) = \sum_{n=-\infty}^{\infty} a(nT)q(t - t_0 - nT) \tag{2.14}$$

where $q(t)$ is a finite energy pulse function. It has been shown in [12] that this $x(t)$ is cyclostationary with period T and has the cyclic auto-correlation and the spectral correlation density given by

$$R_x^\alpha(\tau) = \frac{1}{T}\sum_{l=-\infty}^{\infty} R_a(lT)r_q^\alpha(\tau + lT)e^{-j2\pi\alpha(t_0 + \frac{T}{2}l)} \tag{2.15}$$

$$S_x^\alpha(f) = \frac{1}{T^2}Q(f+\frac{\alpha}{2})Q^*(f-\frac{\alpha}{2})S_a(f+\frac{\alpha}{2})e^{-j2\pi\alpha t_0} \tag{2.16}$$

where

$$r_q^\alpha(\tau + lT) = \frac{1}{T^2}\int_{-\infty}^{\infty} q(t + \frac{\tau + lT}{2} - t_0)q^*(t - \frac{\tau + lT}{2} - t_0)dt \tag{2.17}$$

and $Q(f)$ denotes the Fourier transformation of $q(t)$.

Finally, consider the binary phase shift keying (BPSK) signal;

$$x(t) = b(t)\cos(2\pi f_0 t + \phi_0) \tag{2.18}$$

in which

$$b(t) = \sum_{n=-\infty}^{\infty} b(nT_c)q(t - t_0 - nT) \tag{2.19}$$

where $q(t)$ is a pulse function of width T_c, $1/T_c$ is the keying rate that satisfies $T_c f_c >> 1$, and $b(nT_c)$ is a stochastic binary sequence with values

$$b(nT_c) = \pm 1. \tag{2.20}$$

This BPSK signal is a composition of an AM signal and a PAM signal. It has been shown in [13] that this $x(t)$ is cyclostationary with period T and has the cyclic auto-correlation and the spectral correlation density given by

$$R_x^\alpha(\tau) = R_b^\alpha(\tau)\cos 2\pi\alpha\tau + \frac{1}{2}\left(R_b^{\alpha-2f_0}(\tau)e^{j2\phi_0} + R_b^{\alpha+2f_0}(\tau)e^{-j2\phi_0}\right) \tag{2.21}$$

$$S_x^\alpha(f) = \frac{1}{2}\left(S_b^\alpha(f+f_0) + S_b^\alpha(f-f_0) + S_b^{\alpha-2f_0}(f)e^{j2\phi_0} + S_b^{\alpha+2f_0}(f)e^{-j2\phi_0}\right) \tag{2.22}$$

where $R_b^\alpha(\tau)$ and $S_b^\alpha(f)$ are obtained by replacing T and the subscript a in the right hand sides of (2.15) and (2.16) by T_c and b, respectively.

2.1.3 Discrete-time cyclostationary processes

Here we consider discrete-time cyclostationary processes.

A discrete-time process $x(n)$ with zero mean is said to be cyclostationary with period M if

$$E[\,x(m)x^*(n)\,] = R(m,n) = R(m+M, n+M). \tag{2.23}$$

By changing the variable m in $R(m,n)$ into $n+u$, for fixed u, $R(n+u,n)$ is a periodic sequence in n with period M. Thus we have the following discrete Fourier expansion;

$$R(n+u,n) = \sum_{k=0}^{M-1} c_k(u)W^{-kn} \tag{2.24}$$

where

$$W = e^{-j\frac{2\pi}{M}}. \tag{2.25}$$

Conversely, $c_k(u)$ is expressed as

$$c_k(u) = \frac{1}{M}\sum_{n=0}^{M-1} R(n+u,n)W^{kn}. \tag{2.26}$$

It is stated by Gladyshev [17] that under certain conditions the sequence $c_k(u)$ has the following Fourier representation;

$$c_k(u) = \int_0^{2\pi} F_k(\omega) e^{j\omega u} d\omega \tag{2.27}$$

$$F_k(\omega) = \frac{1}{2\pi} \sum_{u=-\infty}^{\infty} c_k(u) e^{-j\omega u}. \tag{2.28}$$

These $F_k(\omega)$ are also called the spectral correlation density of a discrete-time cyclostationary process. It should be noted that $F_k(\omega)$ is periodic in k with period M and in ω with period 2π, that is,

$$F_{k+Mn}(\omega) = F_k(\omega) \tag{2.29}$$

$$F_k(\omega + 2\pi n) = F_k(\omega) \tag{2.30}$$

for any integer n. Also note that the averaged variance

$$c_0(0) = \frac{1}{M} \sum_{n=0}^{M-1} R(n,n) \tag{2.31}$$

is given by

$$c_0(0) = \int_0^{2\pi} F_0(\omega) d\omega. \tag{2.32}$$

As in the continuous-time case, if $x(n)$ is stationary, then $F_0(\omega)$ reduces to the conventional spectral density and $F_k(\omega) = 0$ for $k = 1, \ldots, M-1$ as follows.

If $x(n)$ is stationary, we have

$$R(m,n) = R_x(m-n) \tag{2.33}$$

where $R_x(u)$ denotes the correlation of the stationary process $x(n)$. Using the identity

$$\sum_{m=0}^{M-1} W^{nm} = \begin{cases} M & \text{for } n : \text{ multiple of } M \\ 0 & \text{otherwise} \end{cases} \tag{2.34}$$

and substituting $R(n+u, n) = R_x(u)$ into (2.26), we have

$$c_k(u) = \begin{cases} R_x(u) & k = 0 \\ 0 & k = 1, \ldots, M-1 \end{cases}, \tag{2.35}$$

then, from (2.28),

$$F_k(\omega) = \begin{cases} F_x(\omega) & k = 0 \\ 0 & k = 1, \ldots, M-1 \end{cases} \tag{2.36}$$

where $F_x(\omega)$ denotes the spectral density of the stationary process $x(n)$.

In the discrete-time case, similarly to the continuous-time case, the cyclic auto-correlation and the spectral correlation density include the auto-correlation and the spectral density as a special case. So, the discrete-time cyclostationary process can be interpreted as a generalization of the discrete-time stationary process.

Now from a cyclostationary process $x(n)$ with period M, define an M-channel process $\boldsymbol{w}(n)$ by

$$\boldsymbol{w}(n) = (w_0(n), w_1(n), \ldots, w_{M-1}(n))^T \tag{2.37}$$

where

$$w_i(n) = x(Mn + i) \quad \text{for } i = 0, \ldots, M-1. \tag{2.38}$$

From

$$\begin{aligned} E[\, w_i(m) w_j^*(n)\,] &= E[\, x(Mm+i)x^*(Mn+j)\,] \\ &= R(Mm+i, Mn+j) \\ &= R(M(m-n)+i, j), \end{aligned} \tag{2.39}$$

the auto-correlation of the M-channel process $\boldsymbol{w}(n)$ is only dependent on the time difference. Therefore the M-channel process $\boldsymbol{w}(n)$ is jointly stationary. Conversely, if $\boldsymbol{w}(n)$ in (2.37) is stationary, it is obvious that $x(n)$ constructed by (2.38) is cyclo-stationary with period M.

The time domain relation between $x(n)$ and $\boldsymbol{w}(n)$ is given by (2.39). In the frequency domain, the relation between $F_k(\omega)$ for $k = 0, 1, \ldots, M-1$ and the spectral density matrix $\boldsymbol{S}(\omega)$ of $\boldsymbol{w}(n)$ is given by

$$\boldsymbol{F}(\omega) = \boldsymbol{V}(\omega)\boldsymbol{S}(M\omega)\boldsymbol{V}^{\dagger}(\omega) \;\; \text{for } |\omega| \le \frac{\pi}{M} \tag{2.40}$$

where $\boldsymbol{F}(\omega)$ is constructed from the spectral correlation density of $x(n)$ by

$$F(\omega) = \begin{pmatrix} F_0(\omega) & F_1^*(\omega+\omega_0) & \cdots & \\ F_1(\omega+\omega_0) & F_0(\omega+\omega_0) & \cdots & \\ \vdots & & \ddots & \cdot \\ F_{M-1}(\omega+(M-1)\omega_0) & \cdots & & F_0(\omega+(M-1)\omega_0) \end{pmatrix} \tag{2.41}$$

for $|\omega| \le \pi/M$ with

$$\omega_0 = \frac{2\pi}{M}, \tag{2.42}$$

$V(\omega)$ is defined by

$$(V(\omega))_{ik} = \frac{1}{\sqrt{M}} e^{-j(\frac{2\pi}{M}ik + k\omega)} \tag{2.43}$$

and $(\cdot)_{ik}$ denotes the (i,k)th element of the matrix. We call the matrix $F(\omega)$ the spectral correlation density matrix. It should be noted that $V(\omega)$ is a unitary matrix.

This relation was first stated by Gladyshev [17] and was presented in this form by Sakai [35]. This is a useful relation because the discrete-time cyclostationary processes can be analyzed by the existing theory and analysis of discrete-time multichannel stationary processes, and vice versa.

We give another different derivation of (2.40) from that by Sakai in [35]. The (i,k)th element of the spectral density matrix is given by

$$(S(\omega))_{ik} = \frac{1}{2\pi} \sum_{\tau=-\infty}^{\infty} (E[\, w(t+\tau) w^{\dagger}(t)\,])_{ik} e^{-j\omega\tau} \quad \text{for } |\omega| \le \pi \tag{2.44}$$

where the dagger denotes the complex conjugate transpose. Substituting

$$(E[\, w(t+\tau) w^{\dagger}(t)\,])_{ik} = R(M(t+\tau)+i, Mt+k) = R(M\tau + i, k) \tag{2.45}$$

into (2.44), we have

$$(S(\omega))_{ik} = \frac{1}{2\pi} \sum_{\tau=-\infty}^{\infty} R(M\tau + i, k) e^{-j\omega\tau}. \tag{2.46}$$

After changing the variable $\tau' = M\tau + i - k$, by using (2.34) and putting $l = i - k$, the above equation reduces to

$$\begin{aligned} (S(\omega))_{ik} &= \frac{1}{2\pi} \sum_{\tau=-\infty}^{\infty} R(k + M\tau + i - k, k) e^{-j\omega\tau} \\ &= \frac{1}{2\pi} \sum_{\tau'=-\infty}^{\infty} R(k+\tau', k) \frac{1}{M} \sum_{m=0}^{M-1} W^{(\tau'-l)m} e^{-j\omega \frac{\tau'-l}{M}}. \end{aligned} \tag{2.47}$$

Also substituting

$$R(k+\tau', k) = \sum_{n=0}^{M-1} c_n(\tau') W^{-nk} \tag{2.48}$$

into (2.47), then we have

$$
\begin{aligned}
(S(\omega))_{ik} &= \frac{1}{2\pi}\sum_{\tau'=-\infty}^{\infty}\sum_{n=0}^{M-1} c_n(\tau')W^{-nk}\frac{1}{M}\sum_{m=0}^{M-1} W^{(\tau'-l)m}e^{-j\omega\frac{\tau'-l}{M}} \\
&= \frac{1}{M}\sum_{m=0}^{M-1}\sum_{n=0}^{M-1} W^{-nk}W^{-lm}e^{j\omega\frac{l}{M}}\frac{1}{2\pi}\sum_{\tau'=-\infty}^{\infty} c_n(\tau')W^{\tau' m}e^{-j\omega\frac{\tau'}{M}} \\
&= \frac{1}{M}\sum_{m=0}^{M-1}\sum_{n=0}^{M-1} W^{(m-n)k-im}e^{j\omega\frac{i-k}{M}}F_n(\frac{\omega+2\pi m}{M}).
\end{aligned}
$$

Changing the variable $n' = m - n$ yields

$$
(S(\omega))_{ik} = \frac{1}{M}\sum_{m=0}^{M-1}\sum_{n'=m-M+1}^{m} W^{n'k-mi}e^{j\omega\frac{i-k}{M}}F_{m-n'}(\frac{\omega+2\pi m}{M}). \tag{2.49}
$$

Noting that

$$
W^{n'+M} = W^{n'}, \quad F_{n'+M-m}(\frac{\omega+2\pi m}{M}) = F_{n'-m}(\frac{\omega+2\pi m}{M}),
$$

we finally have

$$
(S(\omega))_{ik} = \frac{1}{M}\sum_{m=0}^{M-1}\sum_{n'=0}^{M-1} W^{n'k-mi}e^{j\omega\frac{i-k}{M}}F_{m-n'}(\frac{\omega+2\pi m}{M}). \tag{2.50}
$$

This is identical with the (i, k)th element of (2.40). Therefore we also get Gladyshev's relation.

Similarly to the above derivation, if we define an M-channel process $u(n)$ by

$$
u(n) = (u_0(n), u_1(n), \ldots, u_{M-1}(n))^T \tag{2.51}
$$

where

$$
u_i(n) = x(Mn - i) \quad \text{for } i = 0, \ldots, M-1, \tag{2.52}
$$

this M-channel process is jointly stationary. Conversely, if $u(n)$ in (2.51) is stationary, $x(n)$ constructed by (2.52) is cyclostationary with period M. The relation between $F_k(\omega)$ $(k = 0, 1, \ldots, M-1)$ and the spectral density matrix $S(\omega)$ of $u(n)$ in (2.51) is given by

$$
F(\omega) = V^{\dagger}(\omega)S(M\omega)V(\omega). \tag{2.53}
$$

2.1.4 Sampling theorem of cyclostationary processes

In practice, in order to use the digital computer for some processing, the continuous-time processes have to be sampled. Then it is necessary to know the so-called sampling theorem that shows the spectral relation between the original process and the sampled process. The sampling theorem for stationary processes is well known but that for cyclostationary processes is not explicitly shown. So we derive this theorem. For convenience, let us denote a discrete-time sequence with tilde in this section.

Let $x(t)$ be a continuous-time cyclostationary process with period T and the spectral correlation density $S_x^{\alpha}(f)$. Then the sample of $x(t)$ at the time $T_s n$ with the sampling period

$$T_s = \frac{T}{M} \tag{2.54}$$

is given by

$$\tilde{x}(n) = x(T_s n). \tag{2.55}$$

It has been shown in [15] that if $x(t)$ is band-limited $|f| < B$ then

$$S_x^{\frac{i}{T}}(f) = 0, \text{ for } |f| \geq B - |\frac{i}{2T}|. \tag{2.56}$$

The smallest integer M that satisfies

$$\frac{M}{2T} = \frac{1}{2T_s} \geq B \tag{2.57}$$

is selected so that the aliasing does not occur. Note that

$$S_x^{\frac{i}{T}}(f) = 0, \text{ for } |i| \geq M, \tag{2.58}$$

that is, $S_x^{\frac{i}{T}}(f)$ is characterized only at $|i| \leq M-1$. From (2.8), also note that

$$R_x^{\frac{i}{T}}(t) = 0, \text{ for } |i| \geq M. \tag{2.59}$$

From

$$\begin{aligned} R_{\tilde{x}}(m,n) &= E[\, x(T_s m) x^*(T_s n)\,] \\ &= R_x(T_s m, T_s n) \\ &= R_x(T_s m + T, T_s n + T) \\ &= R_x(T_s(m+M), T_s(n+M)) = R_{\tilde{x}}(m+M, n+M), \end{aligned}$$

$\tilde{x}(n)$ is a discrete-time cyclostationary process with period M. Then from (2.26), we have

$$c_k(u) = \frac{1}{M}\sum_{n=0}^{M-1} R_x(T_s(n+u), T_s n) W^{kn}. \tag{2.60}$$

Substituting $t = T_s(n + u/2)$ and $\tau = T_s u$ into (2.5) yields

$$R_x(T_s(n+u), T_s n) = \sum_{i=-\infty}^{\infty} R_x^{\frac{i}{T}}(T_s u) e^{j2\pi \frac{i}{T} T_s (n+\frac{u}{2})}. \tag{2.61}$$

From (2.59), we get

$$R_x(T_s(n+u), T_s n) = \sum_{i=-M+1}^{M-1} R_x^{\frac{i}{T}}(T_s u) e^{j2\pi \frac{i}{T} T_s (n+\frac{u}{2})}. \tag{2.62}$$

Then, substituting this into (2.60) yields

$$\begin{aligned} c_k(u) &= \frac{1}{M}\sum_{n=0}^{M-1}\sum_{i=-M+1}^{M-1} R_x^{\frac{i}{T}}(T_s u) e^{j2\pi \frac{i}{M}(n+\frac{u}{2})} W^{kn} \\ &= \sum_{i=-M+1}^{M-1} R_x^{\frac{i}{T}}(T_s u) e^{j\pi \frac{i}{M} u} \frac{1}{M}\sum_{n=0}^{M-1} e^{j2\pi(\frac{i-k}{M})n}. \end{aligned} \tag{2.63}$$

From (2.34), for $k = 0, \ldots, M-1$, the right hand side of (2.63) is identically zero when $i \neq k$ and $i \neq k - M$. Thus (2.63) reduces to

$$c_k(u) = \begin{cases} R_x^0(T_s u) & \text{for } k = 0 \\ R_x^{\frac{k}{T}}(T_s u) e^{j\pi \frac{k}{M} u} + R_x^{\frac{k-M}{T}}(T_s u) e^{j\pi \frac{k-M}{M} u} & \text{for } k = 1, \ldots, M-1 \end{cases}. \tag{2.64}$$

Substituting this into (2.28) and putting

$$\omega = 2\pi T_s f, \tag{2.65}$$

we have

$$F_k(\omega) = \frac{1}{2\pi}\sum_{u=-\infty}^{\infty} (R_x^{\frac{k}{T}}(T_s u) e^{j\pi \frac{k}{M} u} + R_x^{\frac{k-M}{T}}(T_s u) e^{j\pi \frac{k-M}{M} u}) e^{-j2\pi T_s f u}. \tag{2.66}$$

From (2.7), the first term of the right hand side of (2.66) is given by

$$\frac{1}{2\pi}\sum_{u=-\infty}^{\infty} \left(\int_{-\infty}^{\infty} S_x^{\frac{k}{T}}(\nu) e^{j2\pi\nu T_s u} d\nu\right) e^{j\pi \frac{k}{M} u} e^{-j2\pi T_s f u}. \tag{2.67}$$

After changing the order of summation and integration, using Poisson's summation formula

$$\sum_{\tau=-\infty}^{\infty} e^{j2\pi\mu T \tau} = \frac{1}{T}\sum_{m=-\infty}^{\infty} \delta_a(\mu + \frac{m}{T}) \tag{2.68}$$

where $\delta_a(t)$ denotes Dirac's delta function, we have

$$
\begin{aligned}
&\frac{1}{2\pi}\int_{-\infty}^{\infty} S_x^{\frac{k}{T}}(\nu) \sum_{u=-\infty}^{\infty} e^{j2\pi(\nu+\frac{k}{2T}-f)T_s u} d\nu \\
&= \frac{1}{2\pi T_s}\int_{-\infty}^{\infty} S_x^{\frac{k}{T}}(\nu) \sum_{m=-\infty}^{\infty} \delta_a(\nu + \frac{k}{2T} - f + \frac{m}{T_s}) d\nu \\
&= \frac{1}{2\pi T_s}\sum_{m=-\infty}^{\infty} S_x^{\frac{k}{T}}(f - \frac{k}{2T} - \frac{Mm}{T_s}) \\
&= \frac{1}{2\pi T_s} S_x^{\frac{k}{T}}(f - \frac{k}{2T})
\end{aligned} \tag{2.69}
$$

since from (2.56) the values are identically zero when $m \neq 0$.

Similarly, the second term of the right hand side of (2.66) reduces to

$$
\begin{aligned}
&\frac{1}{2\pi}\int_{-\infty}^{\infty} S_x^{\frac{k-M}{T}}(\nu) \sum_{u=-\infty}^{\infty} e^{j2\pi(\nu+\frac{k-M}{2T}-f)T_s u} d\nu \\
&= \frac{1}{2\pi T_s}\int_{-\infty}^{\infty} S_x^{\frac{k-M}{T}}(\nu) \sum_{m=-\infty}^{\infty} \delta_a(\nu + \frac{k-M}{2T} - f + \frac{m}{T_s}) d\nu \\
&= \frac{1}{2\pi T_s}\sum_{m=-\infty}^{\infty} S_x^{\frac{k-M}{T}}(f - \frac{k-M}{2T} - \frac{Mm}{T}) \\
&= \frac{1}{2\pi T_s} S_x^{\frac{k-M}{T}}(f - \frac{k-M}{2T}).
\end{aligned} \tag{2.70}
$$

Therefore we have

$$
F_k(2\pi T_s f) = \frac{1}{2\pi T_s}(S_x^{\frac{k}{T}}(f - \frac{k}{2T}) + S_x^{\frac{k-M}{T}}(f - \frac{k-M}{2T})). \tag{2.71}
$$

Since from (2.56) the support of $S_x^{\frac{i}{T}}(f)$ at i is given by

$$
-\frac{1}{2T_s} + |\frac{i}{2T}| \leq f \leq \frac{1}{2T_s} - |\frac{i}{2T}| \tag{2.72}
$$

and $k \geq 0$, $S_x^{\frac{k}{T}}(f - k/2T)$ is not identically zero only if

$$
-\frac{1}{2T_s} + \frac{k}{2T} \leq f - \frac{k}{2T} \leq \frac{1}{2T_s} - \frac{k}{2T}. \tag{2.73}
$$

Similarly, $S_x^{\frac{k-M}{T}}(f - (k-M)/2T)$ is not identically zero only if

$$
-\frac{1}{2T_s} - \frac{k-M}{2T} \leq f - \frac{k-M}{2T} \leq \frac{1}{2T_s} + \frac{k-M}{2T}. \tag{2.74}
$$

Thus (2.71) can be rewritten as

$$
F_k(2\pi T_s f) = \begin{cases} \frac{1}{2\pi T_s} S_x^{\frac{k}{T}}(f - \frac{k}{2T}) & \text{for } -\frac{1}{2T_s} + \frac{k}{T} \leq f \leq \frac{1}{2T_s} \\ \frac{1}{2\pi T_s} S_x^{\frac{k-M}{T}}(f - \frac{k-M}{2T}) & \text{for } -\frac{1}{2T_s} \leq f \leq -\frac{1}{2T_s} + \frac{k}{T} \end{cases}. \tag{2.75}
$$

Now in order to see the above relation further, define

$$f_0 = \frac{1}{T}. \tag{2.76}$$

From (2.42), it is noted that

$$\omega_0 = 2\pi T_s f_0 \tag{2.77}$$

and

$$F_k(\omega + \omega_0) = F_k(2\pi T_s(f + f_0)). \tag{2.78}$$

For example, when $M = 2$, Fig. 2.1 and Fig. 2.2 show the support of $S_z^\alpha(f)$ and the corresponding $F_k(\omega) = F_k(2\pi T_s f)$, respectively. The relation (2.75) in the matrix form is given by

$$\begin{pmatrix} F_0(\omega - \omega_0) & F_1(\omega - \omega_0) \\ F_1(\omega) & F_0(\omega) \end{pmatrix} = \frac{1}{2\pi T_s} \begin{pmatrix} S_z^0(f - f_0) & S_z^{-\frac{1}{T}}(f - \frac{f_0}{2}) \\ S_z^{\frac{1}{T}}(f - \frac{f_0}{2}) & S_z^0(f) \end{pmatrix}$$

for $0 \le f \le f_0$ where (i, k) in Fig. 2.1 and Fig. 2.2 are corresponding to the (i, k) elements of the above matrices.

Similarly when $M = 3$, the support of $S_z^\alpha(f)$ and the corresponding $F_k(\omega)$ are shown in Fig. 2.3 and Fig. 2.4. The relation (2.75) in the matrix form is given by

$$\begin{pmatrix} F_0(\omega - \omega_0) & F_2(\omega - \omega_0) & F_1(\omega - \omega_0) \\ F_1(\omega) & F_0(\omega) & F_2(\omega) \\ F_2(\omega + \omega_0) & F_1(\omega + \omega_0) & F_0(\omega + \omega_0) \end{pmatrix}$$
$$= \frac{1}{2\pi T_s} \begin{pmatrix} S_z^0(f - f_0) & S_z^{-\frac{1}{T}}(f - \frac{f_0}{2}) & S_z^{-\frac{2}{T}}(f) \\ S_z^{\frac{1}{T}}(f - \frac{f_0}{2}) & S_z^0(f) & S_z^{-\frac{1}{T}}(f + \frac{f_0}{2}) \\ S_z^{\frac{2}{T}}(f) & S_z^{\frac{1}{T}}(f + \frac{f_0}{2}) & S_z^0(f + f_0) \end{pmatrix}$$

for $|f| \le f_0/2$.

Generally the relation (2.75) is given by

$$\widetilde{F}(\omega) = \widetilde{F}(2\pi T_s f) = S(f) \quad \begin{cases} 0 \le \omega \le \omega_0, \;\; 0 \le f \le f_0 & \text{for } M \text{ is even} \\ |\omega| \le \frac{\omega_0}{2}, \;\; |f| \le \frac{f_0}{2} & \text{for } M \text{ is odd} \end{cases} \tag{2.79}$$

where

$$(\widetilde{F}(\omega))_{ik} = F_{i-k}(\omega - (M' - k)\omega_0) \tag{2.80}$$
$$(S(f))_{ik} = \frac{1}{2\pi T_s} S_z^{\frac{i-k}{T}}(f - (M' - \frac{i+k}{2})f_0) \tag{2.81}$$

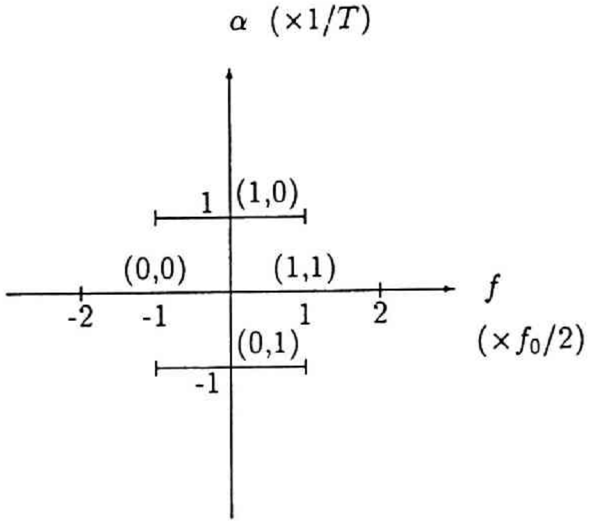

Figure 2.1: The support of $S^{\alpha}(f)$ when $M = 2$.

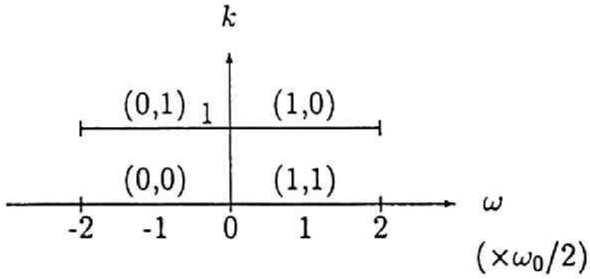

Figure 2.2: The support of $F_k(\omega)$ when $M = 2$.

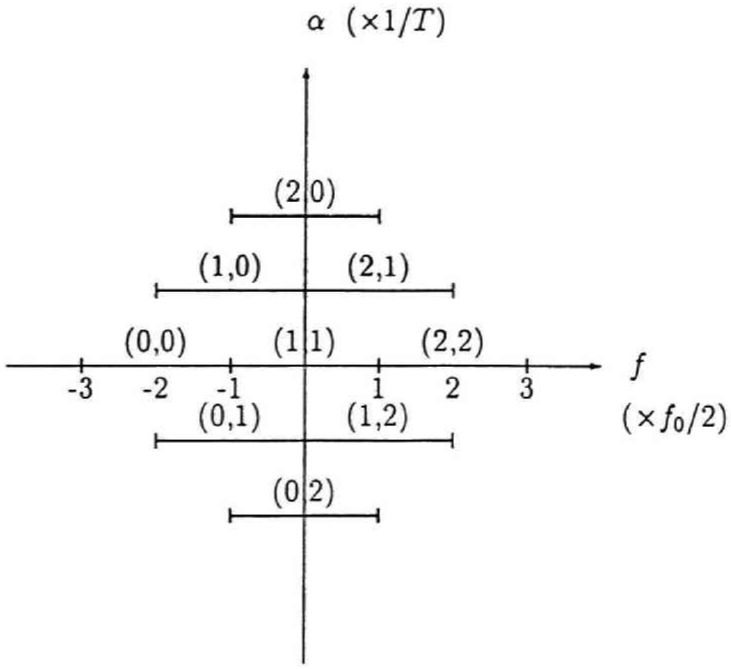

Figure 2.3: The support of $S^{\alpha}(f)$ when $M = 3$.

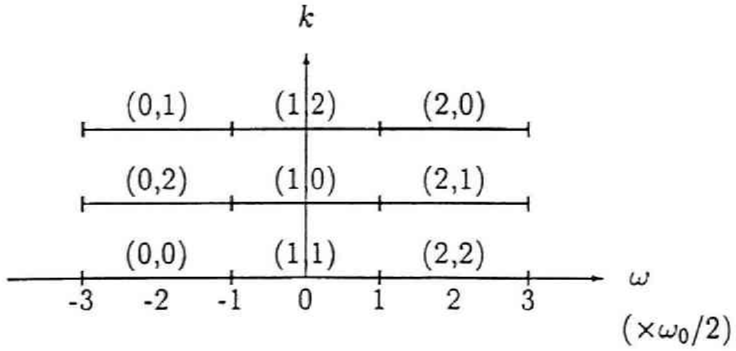

Figure 2.4: The support of $F_k(\omega)$ when $M = 3$.

with

$$M' = \begin{cases} \dfrac{M}{2} & \text{for } M \text{ is even} \\ \dfrac{M-1}{2} & \text{for } M \text{ is odd} \end{cases} . \tag{2.82}$$

This is the matrix version of the sampling theorem of cyclostationary processes.

The fact that the spectral correlation density $F_k(\cdot)$ of the discrete-time process is constructed by those of $S_x^{\frac{k}{T}}(\cdot)$ and $S_x^{\frac{M-k}{T}}(\cdot)$ of the corresponding continuous-time process results from the definition (2.24) where we can not define $R(n+u/2, n-u/2)$ like $R(t+\tau/2, t-\tau/2)$ in the continuous-time case because $u/2$ is not always integer.

As described in Section 2.1.3, it is convenient to construct an M-channel stationary process by

$$\begin{aligned} \tilde{\boldsymbol{x}}(n) &= (\tilde{x}(Mn), \tilde{x}(Mn+1), \ldots, \tilde{x}(Mn+M-1))^T & (2.83) \\ &= (x(Tn), x(Tn+T_s), \ldots, x(Tn+(M-1)T_s)^T. & (2.84) \end{aligned}$$

The spectral relation between $x(t)$ and $\tilde{x}(n)$ and that between $\tilde{x}(n)$ and $\tilde{\boldsymbol{x}}(n)$ are given by (2.79) and (2.53), respectively. In order to obtain the direct spectral relation between $x(t)$ and $\tilde{\boldsymbol{x}}(n)$, define a permutation matrix

$$P_M = \begin{cases} \begin{pmatrix} 0 & I_{M/2} \\ I_{M/2} & 0 \end{pmatrix} & \text{for } M \text{ is even} \\ \begin{pmatrix} 0 & I_{(M-1)/2} \\ I_{(M-1)/2} & 0 \end{pmatrix} & \text{for } M \text{ is odd} \end{cases} \tag{2.85}$$

where I_L denotes the $L \times L$ unit matrix. Then it can be easily shown that the relation $\boldsymbol{F}(\omega)$ in (2.41) and $\widetilde{\boldsymbol{F}}(\omega)$ in (2.80) is given by

$$\boldsymbol{F}(\omega) = \boldsymbol{P}_M \widetilde{\boldsymbol{F}}(\omega) \boldsymbol{P}_M^T \begin{cases} 0 \leq \omega \leq \omega_0 & \text{for } M \text{ is even} \\ |\omega| \leq \dfrac{\omega_0}{2} & \text{for } M \text{ is odd} \end{cases} . \tag{2.86}$$

So, from (2.79) and (2.53), the relation between the spectral correlation density of $x(t)$ and the spectral density matrix of $\tilde{\boldsymbol{x}}(n)$ constructed by (2.83) from $x(t)$, denoted by $\boldsymbol{S}_{\tilde{x}}(\omega)$, is given by

$$\begin{aligned} &\boldsymbol{S}_{\tilde{x}}(M\omega) = \boldsymbol{S}_{\tilde{x}}(2\pi T f) \\ &= \boldsymbol{V}^\dagger(\omega) \boldsymbol{P}_M \boldsymbol{S}(f) \boldsymbol{P}_M^T \boldsymbol{V}(\omega) \begin{cases} 0 \leq \omega \leq \omega_0 & \text{for } M \text{ is even} \\ |\omega| \leq \dfrac{\omega_0}{2} & \text{for } M \text{ is odd} \end{cases} . \end{aligned} \tag{2.87}$$

2.1.5 Estimates of statistics

In many applications, the cyclic auto-correlation and the spectral correlation density are used. But in practice, it is often the case that these statistics can not be explicitly known so that it is necessary to estimate these statistics from the data of finite length. From the sampling theorem, it is sufficient to consider the discrete-time case. We assume that N samples of $x(n)$ from $n = 0$ to $N - 1$ are available.

At first, we give a brief review of the case where the process is stationary with zero mean, the correlation $R(n)$, and the spectral density $F(\omega)$. Next we extend it to the cyclostationary case. We denote the estimate of a value by the same notation with hat.

Usually the estimate of the correlation is given by

$$\widehat{R}(n) = \sum_k w(k)x(k+n)x^*(k) \tag{2.88}$$

where $w(n)$ is a certain window sequence. That of the spectral density based on the above values is given by

$$\widehat{F}(\omega) = \frac{1}{2\pi}\sum_n \widehat{R}(n)e^{-j\omega n}. \tag{2.89}$$

Another estimate of $F(\omega)$ is directly computed from the coefficients of the Discrete Fourier Transformation (DFT) of $x(n)$;

$$X(\omega_k) = \frac{1}{\sqrt{N}}\sum_{n=0}^{N-1} x(n)e^{-j\omega_k n}, \quad \omega_k = \frac{2\pi}{N}k \tag{2.90}$$

by

$$\widehat{F}(\omega_i) = \sum_k w(k)X(\omega_k)X^*(\omega_k). \tag{2.91}$$

The statistical properties of these estimates have been widely studied [18]. It is reported that these estimates are poor when the data length is short. Moreover the window sequence has to be chosen subjectively to obtain good estimates. To avoid these defects, the autoregression (AR) model fitting method was derived to obtain estimates smoother than the method based on the periodgram. In this method the order of the AR model can be evaluated objectively by using the Akaike information criterion (AIC) [1] or some other criteria.

The estimates of continuous-time cyclostationary processes are given and studied in [11] and [15]. Similarly, for discrete-time cyclostationary processes, the estimate of the auto-correlation is given by

$$\widehat{R}(m,n) = \sum_k w(k)x(m+k)x^*(n+k). \tag{2.92}$$

That of the cyclic auto-correlation and the spectral correlation density based on the above values are given by

$$\begin{aligned} \widehat{c}_k(u) &= \frac{1}{M}\sum_{n=0}^{M-1} \widehat{R}(n+u,n)W^{kn} & (2.93)\\ \widehat{F}_k(\omega) &= \frac{1}{2\pi}\sum_{u=-\infty}^{\infty} \widehat{c}_k(u)e^{-j\omega u}. & (2.94)\end{aligned}$$

On the other hand, that of the spectral correlation density at ω_i from the coefficients of the DFT of $x(n)$ can be obtained by

$$\widehat{F}_l(\omega_i) = \sum_k w(k)X(\omega_k)X^*(\omega_k - \frac{2\pi}{M}l). \tag{2.95}$$

for $l = 0, \ldots, M-1$.

Similarly to the stationary case, from the equivalence of the cyclostationary process and the multichannel stationary process, the multivariate AR model fitting method can be used as the estimator of the spectral correlation density. That is, at first, after transforming the cyclostationary process into the multichannel stationary process, the spectral density matrix of this multichannel stationary process can be computed by the multichannel AR model fitting method. Secondly, by using Gladyshev's relation (2.40), the estimate of the spectral correlation density of the original cyclostationary process can be obtained.

2.2 Maximum Likelihood Estimate of TDOA for Cyclostationary Signals

As one application of cyclostationary signal processing, let us consider the problem of estimating the time difference of arrival (TDOA) of an interested signal between two separate sensors spaced h in the presence of uncorrelated noises. The TDOA, denoted by D, is estimated from two received signals at two sensors given by

$$\begin{aligned} y_1(t) &= x(t) + n_1(t) \\ y_2(t) &= x(t-D) + n_2(t) \end{aligned} \tag{2.96}$$

where $x(t)$ is the signal form the source at the first sensor, and $y_i(t)$ and $n_i(t)$ $(i = 1, 2)$ are the received signal and the additive noise at the ith sensor, respectively.

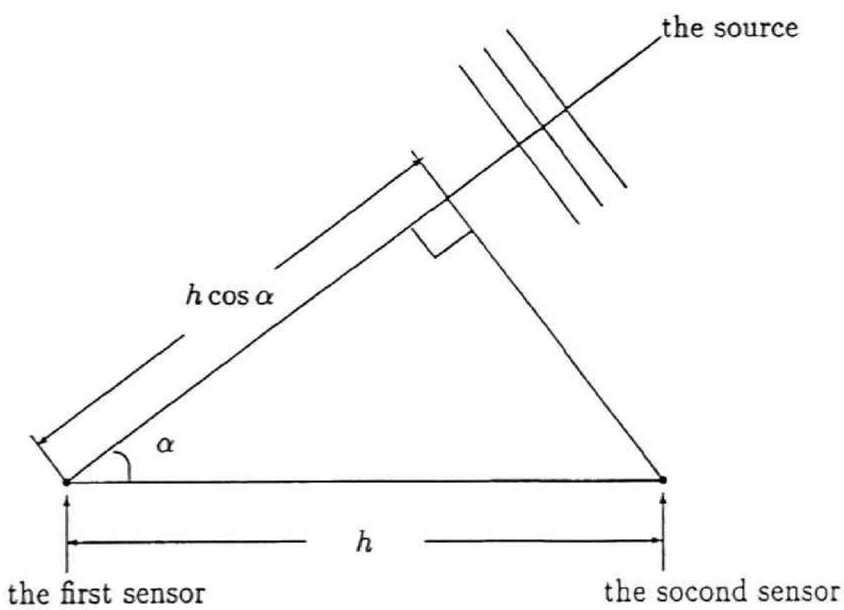

Figure 2.5: Two sensors spaced h receive the signal from the source that is sufficiently far from the sensors with arrival angle α.

See Fig. 2.5 where it is assumed that the source have originated far away from the two sensors. Then, the TDOA can be considered to be equal to the time required for the plane wave to propagate through $h \cos \alpha$ where α denotes the arrival angle of the source. Therefore we get

$$D = \frac{h \cos \alpha}{c} \tag{2.97}$$

where c denotes the velocity of propagation. So, if the velocity of propagation c is known when the estimate $\widehat{D}$ can be obtained, then the estimate of α can be obtained by

$$\widehat{\alpha} = \cos^{-1} \frac{Dc}{h}. \tag{2.98}$$

This estimation problem is one subject in radar and sonar signal processing. Many methods have been derived under the assumption that the source is a stationary signal. But in fact, as seen in Section 2.1.2, many communication signals exhibit cyclostationarity. Gardner and Chen derived the estimator by using explicitly the cyclostationarity of the interested signal [14]. Xu and Kailath also derived another estimator [44]. The former can be considered as an extension of the generalized cross-correlation method [4]. The latter is based on the least squares estimation. In this section, we briefly introduce these methods. It is shown that these methods only use a part of the obtainable information, that is, one slice of the spectral correlation density at a certain α. Then, we derive the maximum likelihood estimator that uses all of the spectral correlation density, which is a generalization of the one obtained by Wax [45] for stationary signals.

2.2.1 Preliminaries

The properties of cyclostationary processes required in this section are at first discussed. We assume that the interested signal $x(t)$ is a continuous-time Gaussian cyclostationary signal with zero mean and period T as treated in Section 2.1.1. Also assume that the noises $n_1(t)$ and $n_2(t)$ are stationary Gaussian noises with zero means and that the signal and the noises are mutually incoherent.

At first, consider

$$y(t) = x(t - D). \tag{2.99}$$

Then, from the definition, the cross-correlation between $y(t)$ and $x(t)$ and the auto-correlation of $y(t)$ are given by

$$R_{yx}(t+\frac{\tau}{2},t-\frac{\tau}{2}) = E[\,y(t+\frac{\tau}{2})x^*(t-\frac{\tau}{2})\,] = R_x(t-D+\frac{\tau}{2},t-\frac{\tau}{2}) \tag{2.100}$$

$$R_y(t+\frac{\tau}{2},t-\frac{\tau}{2}) = E[\,y(t+\frac{\tau}{2})y^*(t-\frac{\tau}{2})\,] = R_x(t-D+\frac{\tau}{2},t-D-\frac{\tau}{2}). \tag{2.101}$$

Substituting (2.100) into (2.6) yields

$$R^{\alpha}_{yx}(\tau) = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} R_x(t-D+\frac{\tau}{2},t-\frac{\tau}{2})e^{-j2\pi\alpha t}dt.$$

After changing the variable $t' = t - D/2$, noting that $R_x(t+\tau/2, t-\tau/2)$ is periodic in t with period T, we have

$$\begin{aligned} R^{\alpha}_{yx}(\tau) &= \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} R_x(t'+\frac{\tau-D}{2},t'-\frac{\tau-D}{2})e^{-j2\pi\alpha(t'+\frac{D}{2})}dt \\ &= R^{\alpha}_x(\tau-D)e^{-j\pi\alpha D} \end{aligned} \tag{2.102}$$

Similarly, substituting (2.101) into (2.6) yields

$$\begin{aligned} R^{\alpha}_{y}(\tau) &= \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} R_x(t-D+\frac{\tau}{2},t-D-\frac{\tau}{2})e^{-j2\pi\alpha t}dt \\ &= \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} R_x(t+\frac{\tau}{2},t-\frac{\tau}{2})e^{-j2\pi\alpha(t+D)}dt \\ &= R^{\alpha}_x(\tau)e^{-j2\pi\alpha D}. \end{aligned} \tag{2.103}$$

Then, by substituting these into (2.7), the spectral correlation densities are given by

$$S^{\alpha}_{yx}(f) = S^{\alpha}_x(f)e^{-j2\pi(f+\frac{\alpha}{2})D} \tag{2.104}$$

$$S^{\alpha}_{y}(f) = S^{\alpha}_x(f)e^{-j2\pi\alpha D}. \tag{2.105}$$

2.2.2 SPECCORR method

Gardner and Chen [14] derive the estimator by using the property of cyclostationary processes (2.104) as follows.

From (2.96) and (2.6), we have

$$R^{\alpha}_{y_1}(\tau) = R^{\alpha}_x(\tau) + R^{\alpha}_{n_1}(\tau) \tag{2.106}$$

where $R^{\alpha}_{y_1}(\tau)$ and $R^{\alpha}_{n_1}(\tau)$ are the cyclic auto-correlation of $y_1(t)$ and $n_1(t)$, respectively.

From (2.10), since the additive noise is stationary, $R^{\alpha}_{n_1}(\tau) = 0$ at any $\alpha \neq 0$. So, we have

$$R^{\alpha}_{y_1}(\tau) = R^{\alpha}_{x}(\tau), \quad \alpha \neq 0. \tag{2.107}$$

Also, from (2.102), we have

$$R^{\alpha}_{y_2 y_1}(\tau) = R^{\alpha}_{x}(\tau - D)e^{-j\pi\alpha D} \tag{2.108}$$

where $R^{\alpha}_{y_2 y_1}(\tau)$ is the cyclic cross-correlation between $y_1(t)$ and $y_2(t)$. The corresponding spectral correlation densities are given by

$$\begin{aligned} S^{\alpha}_{y_1}(f) &= S^{\alpha}_{x}(f) \\ S^{\alpha}_{y_2 y_1}(f) &= S^{\alpha}_{x}(f)e^{-j2\pi(f+\frac{\alpha}{2})D}. \end{aligned}$$

Now considering (2.108) as the system whose input is $R^{\alpha}_{y_1}(\tau)$ and whose output is $R^{\alpha}_{y_1 y_2}(\tau)$, the transfer function of this model is given by

$$H(f) = \frac{S^{\alpha}_{y_2 y_1}(f)}{S^{\alpha}_{y_1}(f)} = e^{-j2\pi(f+\frac{\alpha}{2})D}. \tag{2.109}$$

Then, the impulse response $h(t)$ of this transfer function $H(f)$ is given by

$$\begin{aligned} h(t) &= \int_{-\infty}^{\infty} e^{-j2\pi(f+\frac{\alpha}{2})D} e^{j2\pi f t} df \\ &= e^{-j\pi\alpha D}\delta(t - D). \end{aligned}$$

Obviously, $|h(t)|$ has the maximum value at $t = D$. Using this property, after obtaining the estimates of the spectral correlation densities $\widehat{S}^{\alpha}_{y_2 y_1}(f)$ and $\widehat{S}^{\alpha}_{y_1}(f)$ by a certain method, constructing the estimate of $H(f)$ and the criterion by

$$\widehat{H}(f) = \frac{\widehat{S}^{\alpha}_{y_2 y_1}(f)}{\widehat{S}^{\alpha}_{y_1}(f)} \tag{2.110}$$

$$g(t) := \left| \int_{-\infty}^{\infty} \widehat{H}(f) e^{j2\pi f t} df \right| \tag{2.111}$$

respectively, the estimate of D can be obtained by the value that maximizes the above $g(t)$.

This method is called the SPECCORR method. It should be noted that if $\alpha = 0$ then this method reduces to the conventional GCC method. Also note that this method uses the spectral correlation density only at a certain $\alpha \neq 0$.

2.2.3 Xu-Kailath's method

Xu and Kailath [44] also derive the estimator by using the property of cyclostationary processes (2.105) as follows.

Similarly in (2.107), from (2.103), we have

$$R_{y_2}^{\alpha}(\tau) = R_{y_1}^{\alpha}(\tau)e^{-j2\pi\alpha D}, \quad \alpha \neq 0. \tag{2.112}$$

Consider this equation as the system whose input is $R_{y_1}^{\alpha}(\tau)$ and whose output is $R_{y_2}^{\alpha}(\tau)$. Let us denote the estimates of $R_{y_1}^{\alpha}(\tau)$ and $R_{y_2}^{\alpha}(\tau)$ for $0 \leq \tau \leq \tau_{max}$ computed from a finite number of samples by $\widehat{R}_{y_1}^{\alpha}(\tau)$ and $\widehat{R}_{y_2}^{\alpha}(\tau)$, respectively.

In this case, the coefficients $a = e^{-j2\pi\alpha D}$ can be estimated by the least squares method;

$$\min_{a} \sum_{\tau=0}^{\tau_{max}} \left|\widehat{R}_{y_2}^{\alpha}(\tau) - a\widehat{R}_{y_1}^{\alpha}(\tau)\right|^2.$$

It is well known that the least squares estimate of a is given by

$$\widehat{a} = \left(\sum_{\tau=0}^{\tau_{max}} \left|\left(\widehat{R}_{y_2}(\tau)\right)^{\alpha}\right|^2\right)^{-1} \sum_{\tau=0}^{\tau_{max}} \widehat{R}_{y_2}^{\alpha *}(\tau)\widehat{R}_{y_1}^{\alpha}(\tau).$$

Then, the estimate of D can be obtain by

$$\widehat{D} = -\frac{arg(\widehat{a})}{2\pi\alpha}. \tag{2.113}$$

Since there is no optimization in this method, it is numerically efficient. But from

$$e^{-j2\pi\alpha(D+\frac{k}{\alpha})} = e^{-j2\pi\alpha D} \text{ for any integer } k \tag{2.114}$$

this method can not uniquely determine the estimate unless the value k that satisfies $k/\alpha \leq D < (k+1)/\alpha$ is obtained in advance. It should be noted that this method also uses the cyclic correlation only at a certain $\alpha \neq 0$.

2.2.4 ML method

As mentioned above, the SPECCORR method only uses the information of $S_{y_1y_2}^{\alpha}(f)$ and $S_{y_1}^{\alpha}(f)$ at a certain α and Xu-Kailath's method only uses that of $R_{y_1}^{\alpha}(\tau)$ and $R_{y_2}^{\alpha}(\tau)$ at a certain α, that is, $S_{y_1}^{\alpha}(f)$ and $S_{y_2}^{\alpha}(f)$. Moreover the method of choosing this particular α is not explicitly shown there. So, we derive the ML method that uses all of the information obtained by the received signals.

Assume that we sample the received signals $y_1(t)$ and $y_2(t)$ with the sampling period $T_s = T/M$ as in Section 2.1.4 so that the aliasing does not occur. From the received signals $y_1(t)$ and $y_2(t)$ in (2.96), define the M-channel discrete-time processes as in (2.83) by

$$\boldsymbol{y}_1(n) = (y_1(Tn), y_1(Tn+T_s), \ldots, y_1(Tn+(M-1)T_s))^T \tag{2.115}$$

$$\boldsymbol{y}_2(n) = (y_2(Tn), y_2(Tn+T_s), \ldots, y_2(Tn+(M-1)T_s))^T \tag{2.116}$$

and the joint $2M$-channel discrete-time process by

$$\boldsymbol{y}(n) = \begin{pmatrix} \boldsymbol{y}_1(n) \\ \boldsymbol{y}_2(n) \end{pmatrix}. \tag{2.117}$$

For convenience, we only consider that M is even. But when M is odd, the results can be easily obtain by the similar derivation.

Define

$$\boldsymbol{U}(\omega) = \begin{pmatrix} \boldsymbol{V}(\omega)\boldsymbol{P}_M & 0 \\ 0 & \boldsymbol{V}(\omega)\boldsymbol{P}_M \end{pmatrix} \tag{2.118}$$

with $\boldsymbol{V}(\omega)$ in (2.43), $\boldsymbol{P}_M$ in (2.85), and

$$\boldsymbol{\Lambda} = e^{-\frac{M\pi}{T}D} diag(1, e^{j\frac{2\pi}{T}D}, e^{j\frac{2\pi}{T}2D}, \ldots, e^{j\frac{2\pi}{T}(M-1)D}). \tag{2.119}$$

It is noted that this $\boldsymbol{U}(\omega)$ is also a unitary matrix.

We show that $\boldsymbol{y}(n)$ is $2M$-channel stationary with the spectral density matrix;

$$\boldsymbol{F}_y(M\omega) = \boldsymbol{F}_y(2\pi Tf) = \boldsymbol{U}^\dagger(\omega) \begin{pmatrix} \boldsymbol{S}_{11}(f) & \boldsymbol{S}_{12}(f) \\ \boldsymbol{S}_{21}(f) & \boldsymbol{S}_{22}(f) \end{pmatrix} \boldsymbol{U}(\omega) \tag{2.120}$$

in which

$$\boldsymbol{S}_{11}(f) = \boldsymbol{S}_x(f) + \boldsymbol{S}_{n_1}(f) \tag{2.121}$$

$$\boldsymbol{S}_{21}(f) = e^{-j2\pi fD}\boldsymbol{\Lambda}^\dagger \boldsymbol{S}_x(f) = \boldsymbol{S}_{12}^\dagger(f) \tag{2.122}$$

$$\boldsymbol{S}_{22}(f) = \boldsymbol{\Lambda}^\dagger \boldsymbol{S}_x(f)\boldsymbol{\Lambda} + \boldsymbol{S}_{n_2}(f) \tag{2.123}$$

where $\boldsymbol{S}_x(f)$, $\boldsymbol{S}_{n_1}(f)$ and $\boldsymbol{S}_{n_2}(f)$ denote the spectral correlation density matrix of $x(t)$, $n_1(t)$ and $n_2(t)$ obtained by the same way as in (2.81), respectively.

For $\boldsymbol{S}_{11}(f)$, since the signal and the additive noise are assumed to be mutually incoherent, from (2.81) and (2.96), we have

$$
\begin{aligned}
(\boldsymbol{S}_{11}(f))_{kl} &= \frac{1}{2\pi T_s} S_{y_1}^{\frac{k-l}{T}}\left(f - \frac{M-(k+l)}{2T}\right) \\
&= \frac{1}{2\pi T_s}\left(S_x^{\frac{k-l}{T}}\left(f - \frac{M-(k+l)}{2T}\right) + S_{n_1}^{\frac{k-l}{T}}\left(f - \frac{M-(k+l)}{2T}\right)\right) \\
&= (\boldsymbol{S}_x(f))_{kl} + (\boldsymbol{S}_{n_1}(f))_{kl}.
\end{aligned}
$$

where $S_{y_1}^{\alpha}(f)$, $S_x^{\alpha}(f)$ and $S_{n_1}^{\alpha}(f)$ denote the spectral correlation densities of $y_1(t)$, $x(t)$ and $n_1(t)$. For $\boldsymbol{S}_{21}(f)$, from (2.96) and (2.104),

$$
\begin{aligned}
(\boldsymbol{S}_{21}(f))_{kl} &= \frac{1}{2\pi T_s} S_{y_2 y_1}^{\frac{k-l}{T}}\left(f - \frac{M-(k+l)}{2T}\right) \\
&= \frac{1}{2\pi T_s} S_x^{\frac{k-l}{T}}\left(f - \frac{M-(k+l)}{2T}\right) e^{-j2\pi\left(f - \frac{M-(k+l)}{2T} + \frac{k-l}{2T}\right)D} \\
&= (\boldsymbol{S}_x(f))_{kl} e^{-j2\pi f D} e^{j\frac{2\pi}{T}\left(\frac{M}{2}-k\right)D}
\end{aligned}
$$

where $S_{y_2 y_1}^{\alpha}(f)$ denotes the spectral cross-correlation density between $y_2(t)$ and $y_1(t)$. For $\boldsymbol{S}_{22}(f)$, from (2.105),

$$
\begin{aligned}
(\boldsymbol{S}_{22}(f))_{kl} &= \frac{1}{2\pi T_s}\left(S_{y_2}^{\frac{k-l}{T}}\left(f - \frac{M-(k+l)}{2T}\right) + S_{n_2}^{\frac{k-l}{T}}\left(f - \frac{M-(k+l)}{2T}\right)\right) \\
&= \frac{1}{2\pi T_s} S_x^{\frac{k-l}{T}}\left(f - \frac{M-(k+l)}{2T}\right) e^{-j2\pi\left(\frac{k-l}{T}\right)D} + (\boldsymbol{S}_{n_2}(f))_{kl} \\
&= (\boldsymbol{S}_x(f))_{kl} e^{j\frac{2\pi}{T}(l-k)D} + (\boldsymbol{S}_{n_2}(f))_{kl}
\end{aligned}
$$

where $S_{y_2}^{\alpha}(f)$ and $S_{n_2}^{\alpha}(f)$ denote the spectral correlation densities of $y_2(t)$ and $n_2(t)$. By combining these results and from the sampling theorem of cyclostationary processes (2.87), $\boldsymbol{F}_y(M\omega)$ in (2.120) is obtained.

For notational simplicity, we denote the values of $\boldsymbol{F}_y(M\omega)$, $\boldsymbol{U}(\omega)$ and $\boldsymbol{V}(\omega)$ at $\omega = 2\pi k/MN$ as $\boldsymbol{F}_y(k)$, $\boldsymbol{U}(k)$ and $\boldsymbol{V}(k)$ and denote those of $S_{(\cdot)}(f)$ at $f = M\omega/2\pi T = k/NT$ as $S_{(\cdot)}(k)$.

The DFT computed by N samples of $y(n)$ for $n = 0, \ldots, N-1$ is given by

$$
\boldsymbol{q}(k) = \begin{pmatrix} q_1(k) \\ q_2(k) \end{pmatrix} = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} y(n) e^{-j\nu_k n}, \quad \nu_k = \frac{2\pi}{N} k. \tag{2.124}
$$

It is known that if N is sufficiently large then the Fourier coefficients are complex Gaussian random variables with zero means and the covariance $F_y(k)$ [46]. Then, the ML estimates is given by the value that maximizes

$$\max_{D} \ \log P(q(0), q(1), \ldots, q(N-1)|D)$$

where $P(q(0), q(1), \ldots, q(N-1)|D)$ denotes the joint density function of the Fourier coefficients.

Now we derive the ML estimator. The joint density function is given by

$$P(q(0), q(1), \ldots, q(M-1)|D) = \prod_{k=0}^{N-1} \frac{1}{det(\pi F_y(k))} \exp\left(-q^{\dagger}(k) F_y{}^{-1}(k) q(k)\right). \tag{2.125}$$

Let us define

$$\begin{aligned} \Delta &= S_x(k) + S_{n_2}(k) - S_x(k)\left(S_x(k) + S_{n_1}(k)\right)^{-1} S_x(k) && (2.126) \\ \widetilde{\Delta} &= \Lambda^{\dagger} \Delta(k) \Lambda. && (2.127) \end{aligned}$$

It is noted that Δ does not contain D.

Since $U(\omega)$ and Λ are unitary matrices, the determinant of $F_y(k)$ reduces to

$$\begin{aligned} det F_y(k) &= det \begin{pmatrix} S_{11}(k) & S_{12}(k) \\ S_{21}(k) & S_{22}(k) \end{pmatrix} \\ &= det(S_{22}(k) - S_{21}(k) S_{11}^{-1}(k) S_{12}(k)) \\ &= det\widetilde{\Delta} = det\Delta. \end{aligned} \tag{2.128}$$

This shows that the determinant of $F_y(k)$ does not contain D. Then the logarithm of the likelihood function relevant to D is given by

$$-\sum_{k=0}^{N-1} q^{\dagger}(k) F_y{}^{-1}(k) q(k). \tag{2.129}$$

For each k, from the inverse of the block matrix, we have

$$U(k) F_y{}^{-1}(k) U^{\dagger}(k) = \\ \begin{pmatrix} S_{11}^{-1}(k) + S_{11}^{-1}(k) S_{12}(k) \tilde{\Delta}^{-1}(k) S_{21}(k) S_{11}^{-1}(k) & -S_{11}^{-1}(k) S_{12}(k) \tilde{\Delta}^{-1}(k) \\ -\tilde{\Delta}^{-1}(k) S_{21}(k) S_{11}^{-1}(k) & \tilde{\Delta}^{-1}(k) \end{pmatrix}.$$

From (2.122) and (2.126), we get

$$S_{12}(k)\tilde{\Delta}^{-1}(k)S_{21}(k) = S_z(k)\Delta^{-1}(k)S_z(k).$$

It follows that $S_{12}(k)\tilde{\Delta}^{-1}(k)S_{21}(k)$ is irrelevant to D.

Since $S_{11}(k)$ does not contain D, the term

$$\begin{aligned} &S_{11}^{-1}(k) + S_{11}^{-1}(k)S_{12}(k)\tilde{\Delta}^{-1}(k)S_{21}(k)S_{11}^{-1}(k) \\ &= S_{11}^{-1}(k) + S_{11}^{-1}(k)S_z\Delta^{-1}(k)S_z(k)S_{11}^{-1}(k) \end{aligned}$$

does not contain D either.

From the above discussion, substituting (2.121)~(2.123) into (2.129), the logarithm of the likelihood function with the terms relevant to D is

$$\sum_{k=0}^{N-1} \{2\mathrm{Re}\left(h_1(k,t)e^{j\frac{\nu_k}{T}t}\right) - h_2(k,t)\} \tag{2.130}$$

where

$$\begin{aligned} h_1(k,t) &= q_1^\dagger(k)P_M V^\dagger(k)G(k)\Lambda V(k)P_M q_2(k) \\ h_2(k,t) &= q_2^\dagger(k)P_M V^\dagger(k)\Lambda^\dagger\Delta^{-1}(k)\Lambda V(k)P_M q_2(k) \end{aligned}$$

and

$$G(k) = \left(S_z(k) + S_{n_1}(k)\right)^{-1} S_z(k)\Delta^{-1}(k). \tag{2.131}$$

Therefore, by maximizing (2.130) with respect to D, the ML estimate can be obtained.

In practice, to compute this criterion, it is necessary to obtain the estimates in (2.126) and (2.131). The spectral density matrix $F_y(k)$ can be estimated by a certain method. And $S_z(k) + S_{n_1}(k)$ is $S_{11}(k)$ in $F_y(k)$. For $S_z(k)$ in $S_{21}(k)$ and $S_z(k) + S_{n_2}(k)$ in $S_{22}(k)$, since $S_{21}(k)$ and $S_{22}(k)$ involve the unknown value D, these estimates are constructed as follows.

Once $F_y(k)$ is estimated, from (2.120), $S_{11}(k)$, $S_{21}(k)$ and $S_{22}(k)$ can be obtained. Since the additive noise is assumed to be stationary, the non-diagonal elements of $S_z(k) + S_{n_1}(k)$ are constructed only by $S_z(k)$ as

$$(S_z(k))_{ij} = (S_{11}(k))_{ij}, \quad i \neq j. \tag{2.132}$$

Therefore the estimates of the non-diagonal elements of $S_{11}(k)$ can be used as those of $S_x(k)$.

Also since the diagonal elements of $S_x(k)$ have non-negative value and Λ is a diagonal matrix that satisfies

$$|(\Lambda)_{ii}| = 1,$$

we have

$$(S_x(k))_{ii} = |(S_{21}(k))_{ii}|. \tag{2.133}$$

Thus the absolute value of the diagonal elements of $S_{21}(k)$ can be used as the estimates of the diagonal elements of $S_x(k)$. From (2.132) and (2.133), all the elements of $S_x(k)$ can be obtained.

Similarly, the estimates of the non-diagonal elements of $S_x(k)$ can be used as those of $S_x(k) + S_{n_2}(k)$.

From

$$(S_x(k) + S_{n_2}(k))_{ii} = (S_{22}(k))_{ii},$$

the estimates of the diagonal elements of $S_{22}(k)$ can be used as those of $S_x(k) + S_{n_2}(k)$. From these values, all the elements of $S_x(k) + S_{n_2}(k)$ can be obtained.

2.2.5 Numerical results

By the methods as discussed in Section 2.1.5, we first estimate the spectral correlation density of the received signals. Fig. 2.6 and Fig. 2.7 show examples of the magnitude of the estimates of the spectral correlation density $S_{y_1}^{0}(f)$ and $S_{y_1}^{1/T}(f)$ of an AM signal with the carrier frequency $f_c = 0.25/T_s$ and the band width $0.04/T_s$ where $M = 2$, $T = 2T_s$, $D = 10T_s$, the total number of samples is 512, and the additive noises are white Gaussian. The signal/noise ratio (SNR) defined by

$$SNR = 10 \log \frac{\sigma_x^2}{\sigma_{n_1}^2} \tag{2.134}$$

is -2dB where σ_x^2 and $\sigma_{n_1}^2$ are the variances of the signal and the noise, respectively.

The estimates by the method based on the periodgram are smoothed by 30 samples of coefficients of the DFT of the received signal. The multichannel AR model fitting method is computed by the LWR algorithm [47] using the AIC criterion for the order estimation. It can be said that the estimate of the multichannel AR model fitting method is better than that of the method based on the periodgram.

Fig. 2.8 shows the histograms of 100 estimates of the ML method and those of the SPECCORR method for the above AM signal where the total number of samples is 512, $D = 10T_s$ and the SNR is 38dB. The horizontal axis denotes $\widehat{D}$ and the vertical axis denotes the frequencies of occurrence. Here, the estimates of the spectral correlation density needed in the estimation are obtained by the multichannel AR model fitting method. Computing each criterion requires nonlinear optimization, so the search is done in the range $[5T_s, 15T_s]$ with the step size $T_s/10$. As shown in Fig. 2.9, in the ML method its estimates are around the true value, but in the SPECCORR method those are not.

Table. 2.1 shows that the empirical mean and variances of these methods. The bias of ML method is smaller than that of the SPECCORR method. This is especially true at low SNR.

However, in the ML method, the criterion function is quasi periodic with period $1/f_c$ as shown in Fig. 2.9. Since the difference between the peak value of the main lobe and those of the side lobes is small, it is often the case that the peak value of the lobe containing the true value is smaller than those of other lobes. Therefore, a few estimates are around $D + n/f_c$ for a certain integer n. These estimates obtained in the wrong lobe badly contribute to the mean squared error. This phenomena is also reported in [45] that treats the ML estimates when the interested sources is stationary.

Table. 2.2 shows that the empirical means and variances of the ML estimate when the range is restricted to $[7T_s, 13T_s]$. That is, the search is done only in the lobe containing the true value. It can be seen that the ML method can obtain the good estimates. So it can be said that the ML method is effective when it is known that D is small, that is, it is sufficient to search only around 0. Or after a estimate is obtained by other methods, by using the ML method around this estimate, better estimate can be obtained.

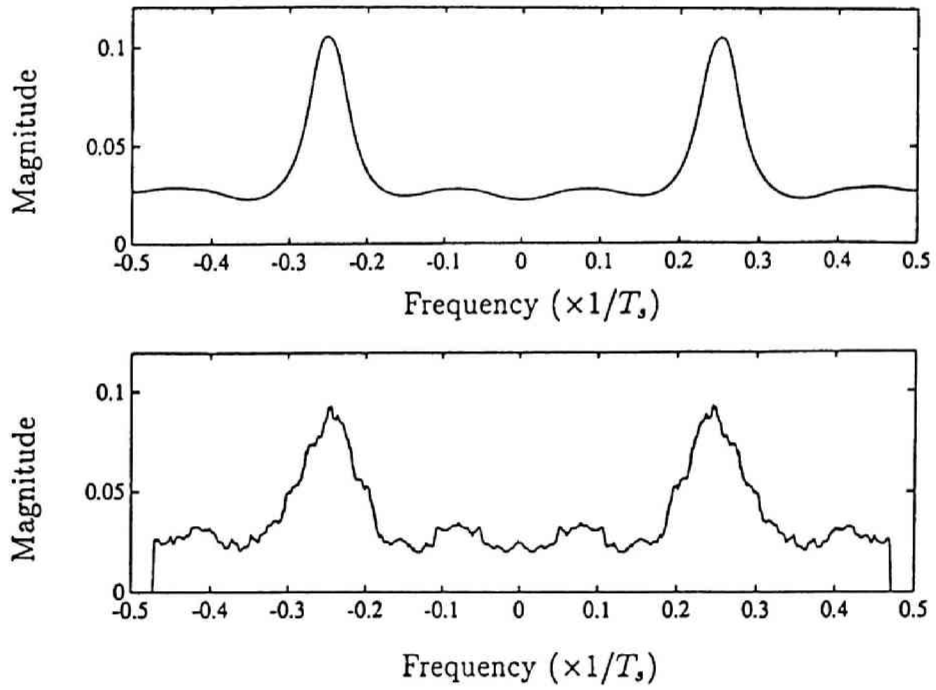

Figure 2.6: Magnitude of estimated spectral correlation density $S^0_{y_1}(f)$, (the above by AR, the below by periodgram).

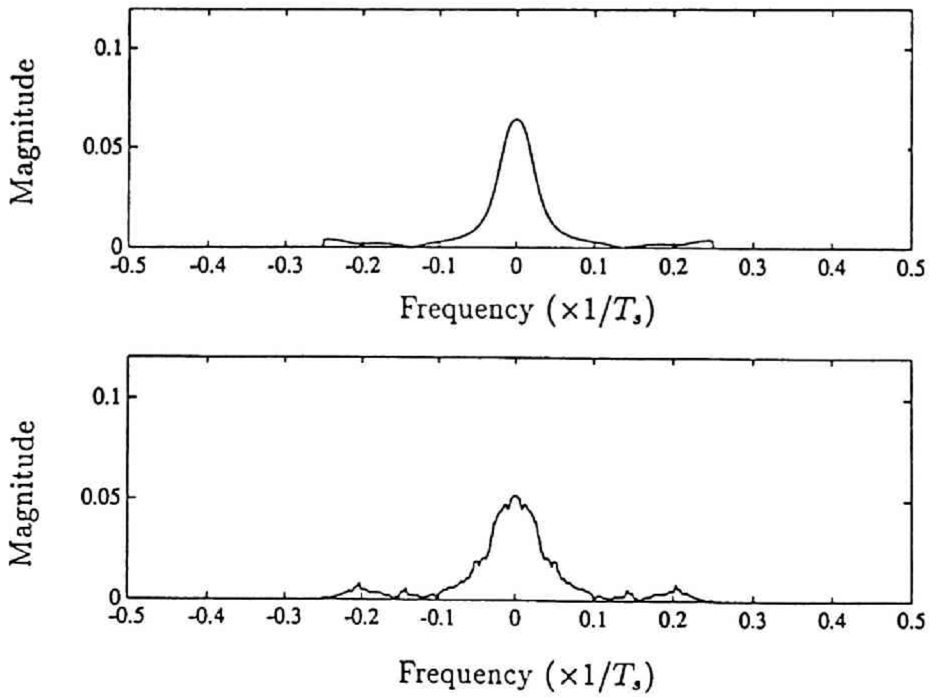

Figure 2.7: Magnitude of estimated spectral correlation density $S^{\frac{1}{T}}_{y_1}(f)$, (the above by AR, the below by periodgram).

	ML	
SNR	Mean	Cov.
38	9.914	0.555
18	10.084	1.888
-2	10.122	7.871

	SPECCORR	
SNR	Mean	Cov.
38	9.446	1.011
18	7.951	6.967
-2	7.597	12.162

Table 2.1: Empirical means and variances of the ML estimate and the SPECCORR.

	ML	
SNR	Mean	Cov
38	9.961	0.079
18	10.012	0.040
-2	10.047	0.382

Table 2.2: Empirical means and variances of the ML estimate when the searching range is restricted.

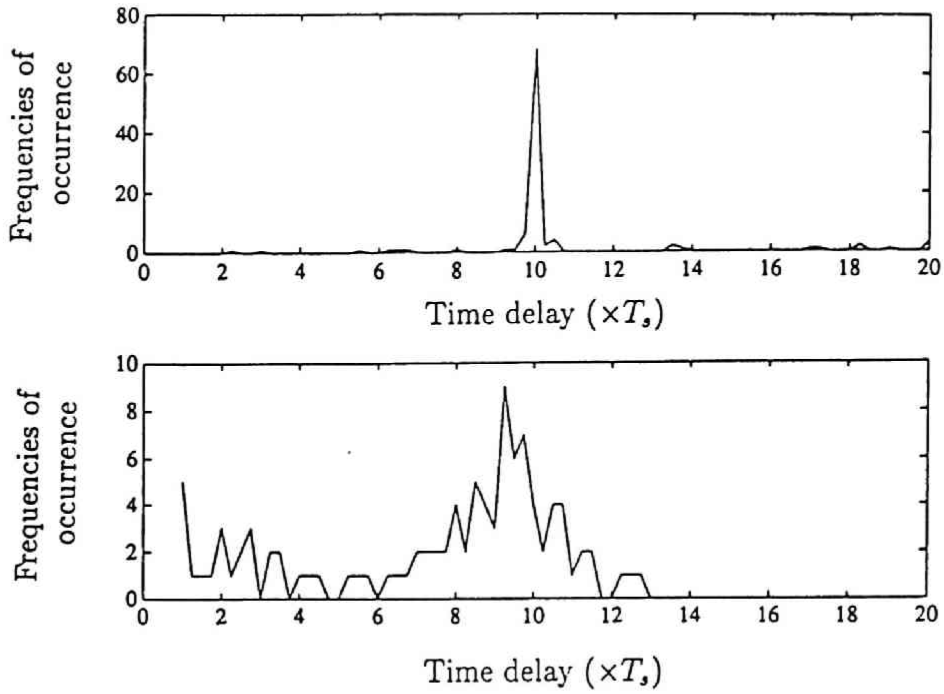

Figure 2.8: Histogram of TDOA estimates, (the above by AR, the below by periodogram).

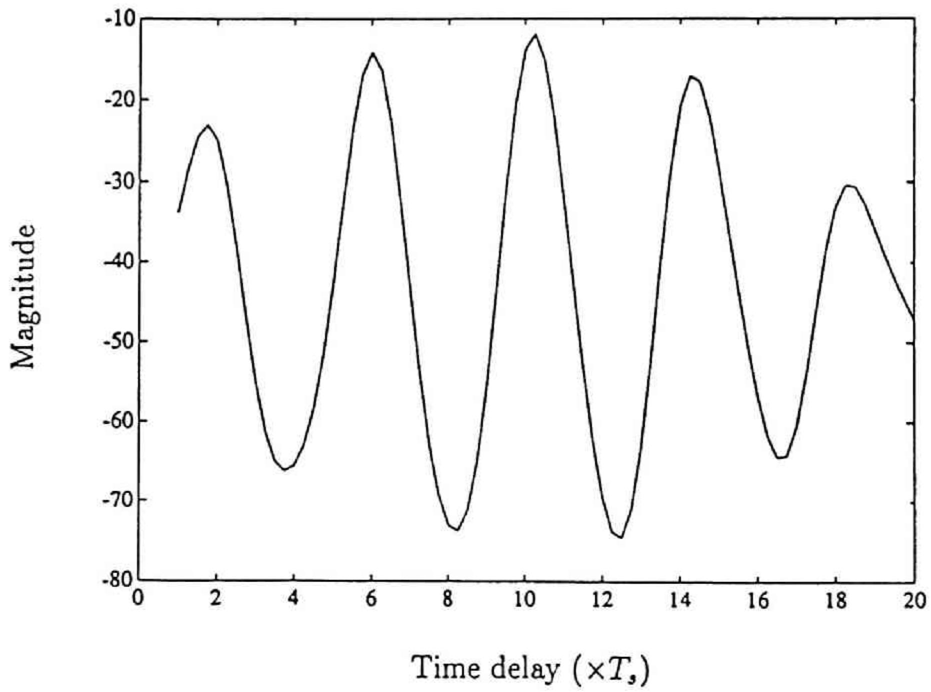

Figure 2.9: Sample values of the criterion.

Chapter 3

OPTIMIZATION OF FILTER BANKS USING CYCLOSTATIONARY SPECTRAL ANALYSIS

This chapter studies filter banks in multirate systems. In Section 3.1, multirate systems are reviewed under the assumption that the treated signals are deterministic l^2 sequences. In Section 3.2, the theory of multirate systems for stochastic signals are considered. Since the outputs of multirate systems usually exhibit the cyclostationarity when the input signals are considered to be stochastic signals, the spectral correlation density of the output of M-band filter banks are derived. It is proven that the output of an alias free filter bank for any stationary input is stationary. In Section 3.3, the perfect reconstruction filter banks are briefly introduced. By the cyclostationary spectral analysis, in Section 3.4, we derive a criterion for minimization of reconstruction error when some subband signals are dropped. This is used to construct the optimal biorthogonal filter banks, the conjugate quadrature filter banks and the perfect reconstruction linear phase filter banks. We also evaluate the obtained optimal filter banks from other criteria.

3.1 Fundamentals of Multirate Systems

In this chapter, we only treat filters with real coefficients and follow the notations in [40]. Now we briefly review the fundamentals of multirate systems. The results are derived under the assumption that the interested sequences are l_2 deterministic signals. For simplicity of analysis, although there are quantizing operations, we ignore the quantizing errors. Let us define the z-transform of a sequence $h(n)$ by

$$H(z) = \sum_{n=0}^{\infty} h(n) z^{-n}. \tag{3.1}$$

Decimator Interpolator

Figure 3.1: Decimator and interpolator.

The M-fold decimation in the left hand side of Fig. 3.1 is to produce the output sequence

$$y(n) = x(Mn) \tag{3.2}$$

for an input sequence $x(n)$ where M is an integer. Only those samples of $x(n)$ that occur at time equal to multiples of M are retained. In the transform domain, it can be expressed by

$$Y(z) = \frac{1}{M} \sum_{k=0}^{M-1} X(z^{\frac{1}{M}} W^k) := X(z)|_{\downarrow M} \tag{3.3}$$

where

$$W = e^{-j\frac{2\pi}{M}} \tag{3.4}$$

and $X(z)$ and $Y(z)$ are the z-transforms of $x(n)$ and $y(n)$, respectively.

The L-fold interpolation in the right hand side of Fig. 3.1 is to produce the output sequence

$$y(n) = \begin{cases} x(\frac{n}{L}) & n : \text{multiple of } L \\ 0 & \text{otherwise} \end{cases} \tag{3.5}$$

with an integer L. The output is obtained by interpolating $L-1$ zeros between the input signal. In the transform domain, it can be expressed by

$$Y(z) = X(z^L) := X(z)|_{\uparrow L}. \tag{3.6}$$

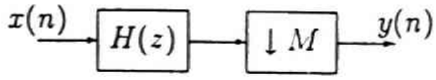

Figure 3.2: Decimation filter.

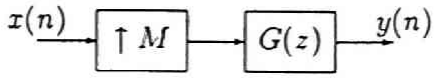

Figure 3.3: Interpolation filter.

Fig. 3.2 shows the decimation filter that is followed by the decimator to reduce the aliasing. The input-output relation of this system is given by

$$
\begin{aligned}
y(n) &= \sum_k h(k)x(Mn-k) &\quad (3.7)\\
&= \sum_k h(Mn-k)x(k). &\quad (3.8)
\end{aligned}
$$

Fig. 3.3 shows the interpolation filter that is preceded by the interpolator to reduce the imaging. The input-output relation of this system is given by

$$
\begin{aligned}
y(n) &= \sum_k g(k)x(n-Lk) &\quad (3.9)\\
&= \sum_k g(n-Lk)x(k). &\quad (3.10)
\end{aligned}
$$

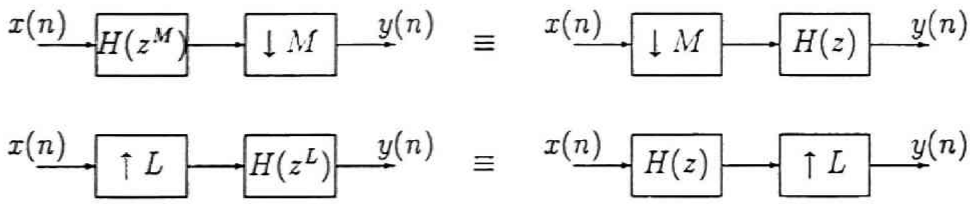

Figure 3.4: Noble identities.

See the upper left hand side of Fig. 3.4. The input-output relation is given by

$$y(n) = \sum_k h(k)x(M(n-k)). \quad (3.11)$$

Changing the variable in the above equation yields

$$y(n) = \sum_k h(n-k)x(Mk) \quad (3.12)$$

which is the input-output relation of the upper right hand side of Fig. 3.4. Thus the system of the upper left hand side of Fig. 3.4 and that of the upper right hand side of Fig. 3.4 are equivalent. In the transform domain, it can be expressed as

$$\begin{aligned} Y(z) &= (H(z^M)X(z))|_{\downarrow M} && (3.13)\\ &= H(z)(X(z)|_{\downarrow M}). && (3.14) \end{aligned}$$

Similarly, the input-output relation of the lower left hand side of Fig. 3.4 is given by

$$y(n) = \begin{cases} \sum_k h(k)x(\frac{n}{L} - k) & n : \text{multiple of } L \\ 0 & \text{otherwise} \end{cases} . \qquad (3.15)$$

Changing the variable in the above equation yields

$$y(n) = \begin{cases} \sum_k h(\frac{n}{L} - k)x(k) & n : \text{multiple of } L \\ 0 & \text{otherwise} \end{cases} \qquad (3.16)$$

which is the input-output relation of the lower right hand side of Fig. 3.4. Thus the system of the lower left hand side of Fig. 3.4 and that of the lower right hand side of Fig. 3.4 are equivalent. In the transform domain, it can be expressed as

$$\begin{aligned} Y(z) &= H(z^L)(X(z)|_{\uparrow L}) && (3.17)\\ &= (H(z)X(z))|_{\uparrow L}. && (3.18) \end{aligned}$$

These relation are known as the noble identities [40]. These are useful in multirate systems.

Separating the coefficients $h(n)$ of $H(z)$ in terms of n modulo M, we can write

$$H(z) = \sum_{l=0}^{M-1} z^{-l} E_l(z^M) \qquad (3.19)$$

where

$$E_l(z) = \sum_{n=-\infty}^{\infty} h(Mn + l)z^{-n}, \quad 0 \leq l \leq M - 1 .$$

Similarly $H(z)$ can be written as

$$H(z) = \sum_{l=0}^{M-1} z^{-(M-1-l)} R_l(z^M) \qquad (3.20)$$

with

$$R_l(z) = E_{M-1-l}(z) \;\; 0 \le l \le M-1 \;\; . \tag{3.21}$$

(3.19) and (3.20) are called Type 1 and Type 2 polyphase representation of $H(z)$, respectively [40]. These play a great role in multirate systems.

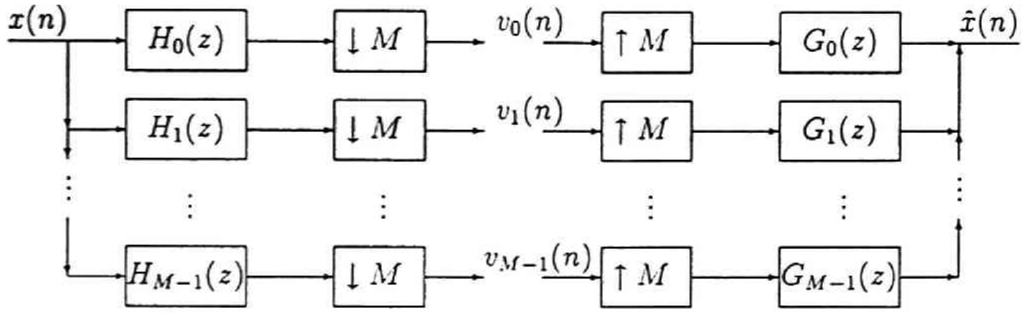

Figure 3.5: M-band filter bank.

Now we consider the M-band filter bank shown in Fig. 3.5. The left hand side and the right hand side of the filter bank are called the analysis bank and the synthesis bank, respectively. The signals processed by the analysis bank are called the subband signals.

Define analysis filters and synthesis filters in the M-band filer bank in Fig. 3.5 as

$$\begin{aligned} \boldsymbol{H}(z) &= (H_0(z), H_1(z), \ldots, H_{M-1}(z))^T & (3.22) \\ \boldsymbol{G}(z) &= (G_0(z), G_1(z), \ldots, G_{M-1}(z))^T & (3.23) \end{aligned}$$

respectively. Using Type 1 polyphase representations of analysis filters and synthesis filters

$$\begin{aligned} H_k(z) &= \sum_{l=0}^{M-1} z^{-l} E^h_{kl}(z^M) & (3.24) \\ G_k(z) &= \sum_{l=0}^{M-1} z^{-l} E^g_{kl}(z^M), & (3.25) \end{aligned}$$

we can write

$$\begin{aligned} \boldsymbol{H}(z) &= \boldsymbol{E}_h(z^M)\boldsymbol{e}(z) & (3.26) \\ \boldsymbol{G}(z) &= \boldsymbol{E}_g(z^M)\boldsymbol{e}(z) & (3.27) \end{aligned}$$

where

$$e(z) = \left(1, z^{-1}, \ldots, z^{-M+1}\right)^T \tag{3.28}$$

and

$$\begin{aligned}(\boldsymbol{E}_h(z))_{kl} &= E^h_{kl}(z) &(3.29)\\ (\boldsymbol{E}_g(z))_{kl} &= E^g_{kl}(z). &(3.30)\end{aligned}$$

Using the noble identities (3.14), we interchange the filters by the decimator and the interpolator respectively so that Fig. 3.6 can be obtained.

It should be noted that in [40] Type 2 polyphase representation of $\boldsymbol{G}(z)$ is used as shown in Fig. 3.7 where

$$\boldsymbol{G}(z) = \boldsymbol{R}_g^T(z^M)\left(z^{-M+1}, \ldots, z^{-1}, 1\right)^T \tag{3.31}$$

and

$$\boldsymbol{R}_g^T(z) = \boldsymbol{E}_g(z) J_M \tag{3.32}$$

with the $M \times M$ matrix

$$J_M = \begin{pmatrix} 0 & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & 0 \end{pmatrix}. \tag{3.33}$$

The alias component (AC) matrix of a filter bank $H(z)$ are defined by

$$\boldsymbol{H}_{AC}(z) = \begin{pmatrix} H_0(z) & H_1(z) & \cdots & H_{M-1}(z) \\ H_0(zW) & H_1(zW) & \cdots & H_{M-1}(zW) \\ \vdots & \vdots & \ddots & \vdots \\ H_0(zW^{M-1}) & H_1(zW^{M-1}) & \cdots & H_{M-1}(zW^{M-1}) \end{pmatrix}. \tag{3.34}$$

This also plays a great role in multirate systems. The relation between the AC matrix and the polyphase matrix is given by

$$\begin{aligned}\boldsymbol{H}_{AC}^T(z) &= \left(\boldsymbol{H}(z), \boldsymbol{H}(zW), \ldots, \boldsymbol{H}(zW^{M-1})\right)\\ &= \boldsymbol{E}_h(z^M)\left(e(z), e(zW), \ldots, e(zW^{M-1})\right)\\ &= \boldsymbol{E}_h(z^M)\boldsymbol{\Lambda}(z)\boldsymbol{W}^\dagger &(3.35)\end{aligned}$$

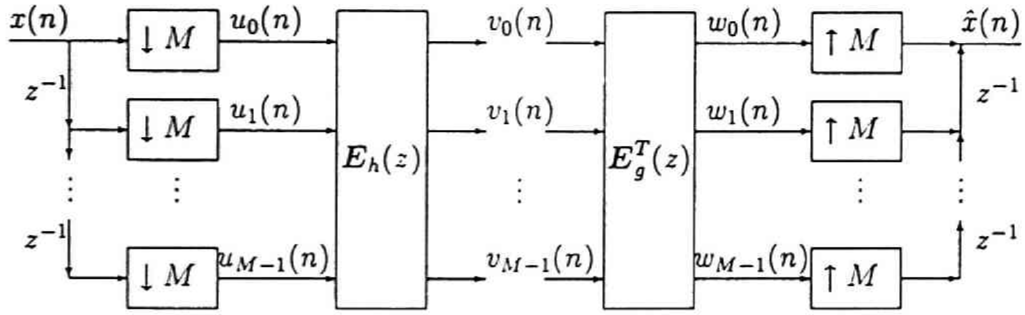

Figure 3.6: M-band filter bank by using Type 1 polyphase matrices.

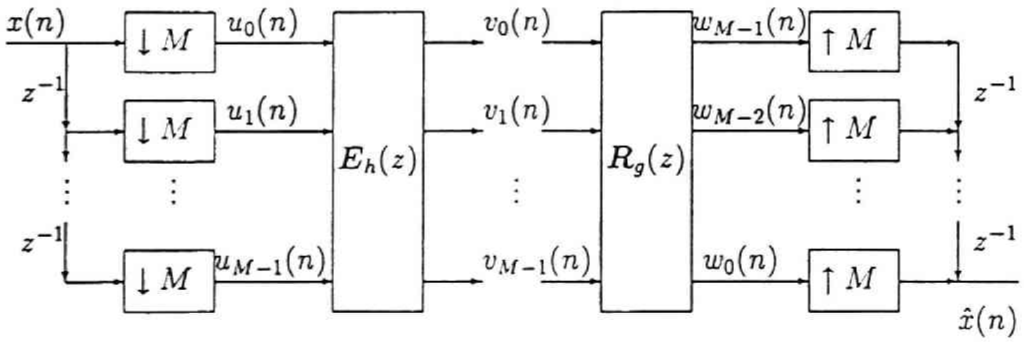

Figure 3.7: M-band filter bank by using Type 1 polyphase matrix in the analysis bank and Type 2 polyphase matrix in the synthesis bank.

where $\boldsymbol{W}$ is the DFT matrix such that

$$\boldsymbol{W} = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & W & \cdots & W^{M-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & W^{M-1} & \cdots & W^{(M-1)(M-1)} \end{pmatrix} \tag{3.36}$$

and

$$\boldsymbol{\Lambda}(z) = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & z^{-1} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & z^{-M+1} \end{pmatrix}. \tag{3.37}$$

Define the AC matrix of the synthesis filters by $\boldsymbol{G}(z)$ as $\boldsymbol{G}_{AC}(z)$ by the same way.

By using the AC matrix, the input-output relation of the M-band filter bank in Fig. 3.5 is given by

$$\widehat{X}(z) = \frac{1}{M}\boldsymbol{X}^T(z)\boldsymbol{H}_{AC}(z)\boldsymbol{G}(z) \tag{3.38}$$

where $\widehat{X}(\mathrm{z})$ denotes the z-transform of $\widehat{x}(n)$ and

$$\boldsymbol{X}(z) = \left(X(z), X(zW), \ldots, X(zW^{M-1})\right)^T. \tag{3.39}$$

The filter bank is said to be alas free if

$$\widehat{X}(z) = T(z)X(z), \tag{3.40}$$

that is, $\widehat{X}(z)$ has no aliasing terms $X(zW), \ldots, X(zW^{M-1})$. It should be noted that the alias free filter bank reduces to the linear time-invariant (LTI) system whose transfer function $T(z)$ is given by

$$\frac{1}{M}\boldsymbol{H}_{AC}(z)\boldsymbol{G}(z) = (T(z), 0, \ldots, 0)^T. \tag{3.41}$$

It has been shown in [40] that a filter bank is alias free if and only if

$$\boldsymbol{P}(z) := \boldsymbol{R}_g(z)\boldsymbol{E}_h(z) = \begin{pmatrix} p_0(z) & p_1(z) & \cdots & p_{M-2}(z) & p_{M-1}(z) \\ z^{-1}p_{M-1}(z) & p_0(z) & \cdots & p_{M-3}(z) & p_{M-2}(z) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ z^{-1}p_1(z) & z^{-1}p_2(z) & \cdots & z^{-1}p_{M-1}(z) & p_0(z) \end{pmatrix} \tag{3.42}$$

holds.

Moreover, it is also shown in [40] that a filter bank is said to be a perfect reconstruction (PR) filter bank if

$$\hat{x}(n) = cx(n-n_0), \quad n_0 : \text{integer}, \tag{3.43}$$

for some integer n_0 and some constant $c \neq 0$. The necessary and sufficient condition for this is

$$\boldsymbol{P}(z) = \boldsymbol{R}_g(z)\boldsymbol{E}_h(z) = cz^{-m_0}\begin{pmatrix} 0 & I_{M-r} \\ z^{-1}I_r & 0 \end{pmatrix} \tag{3.44}$$

for some integer r with $0 \leq r \leq M-1$ and $Mm_0 = n_0 - r - M + 1$ [40]. That is,

$$p_k(z) = \begin{cases} cz^{-m_0} & \text{for } k = r \\ 0 & \text{otherwise,} \end{cases} \tag{3.45}$$

and

$$T(z) = cz^{-M}z^{-(r-1)}z^{-Mm_0} = cz^{-n_0}. \tag{3.46}$$

3.2 Output of Filter Banks from the Stochastic Point of View

In this section, we consider filter banks from the stochastic point of view by using the cyclostationary analysis since the output of a filter bank for a stationary input is generally cyclostationary. We derive the spectral correlation density matrix $\boldsymbol{F}(\omega)$ of the output $\hat{x}(n)$ of the M-band filter bank in Fig. 3.6 when the input $x(n)$ is a cyclostationary process with zero mean, period M, and the spectral correlation density matrix $\boldsymbol{F}_x(\omega)$.

At first, Gladyshev's relations (2.40) and (2.53) are summarized in terms of multirate systems so that they can be easily used.

In (2.38), we put

$$w_i(n) = x(Mn+i) \quad \text{for } i = 0, \ldots, M-1.$$

If $x(n)$ is input and $w_i(n)$ is output,then (2.38) can be written as the left hand side of Fig. 3.8. Conversely, if $w_i(n)$ is input and $x(n)$ is output, then this can be written

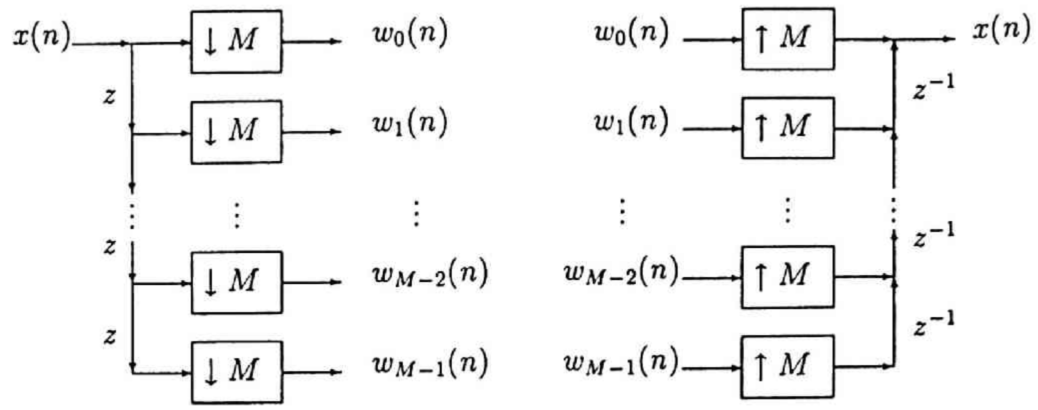

Figure 3.8: Figure of the relation $w_i(n) = x(Mn + i)$.

as the right hand side of Fig. 3.8. By using $\boldsymbol{W}$ in (3.36) and $\boldsymbol{\Lambda}(z)$ in (3.79), $\boldsymbol{V}(\omega)$ in (2.43) is written by

$$\boldsymbol{V}(\omega) = \frac{1}{\sqrt{M}} \boldsymbol{W} \boldsymbol{\Lambda}(e^{j\omega}). \tag{3.47}$$

Then, Gladyshev's relation (2.40) can be rewritten as

$$\boldsymbol{F}_x(\omega) = \frac{1}{M} \boldsymbol{W} \boldsymbol{\Lambda}(e^{j\omega}) \boldsymbol{S}_w(M\omega) \boldsymbol{\Lambda}(e^{-j\omega}) \boldsymbol{W}^\dagger \quad \text{for } |\omega| \leq \frac{\pi}{M} \tag{3.48}$$

where $\boldsymbol{S}_w(M\omega)$ is the spectral density matrix of $\boldsymbol{w}(n)$ constructed from $w_i(n)$ in (2.37) by

$$\boldsymbol{w}(n) = (w_0(n), w_1(n), \ldots, w_{M-1}(n))^T.$$

Similarly, in (2.52), we put

$$u_i(n) = x(Mn - i) \quad \text{for } i = 0, \ldots, M-1.$$

If $x(n)$ is input and $u_i(n)$ is output, then (2.52) can be written as the left hand side of Fig. 3.9. Conversely, if $u_i(n)$ is input and $x(n)$, then this can be written as the right hand side of Fig. 3.9. Gladyshev's relation (2.53) can be rewritten as

$$\boldsymbol{F}_x(\omega) = \frac{1}{M} \boldsymbol{W}^\dagger \boldsymbol{\Lambda}(e^{-j\omega}) \boldsymbol{S}_u(M\omega) \boldsymbol{\Lambda}(e^{j\omega}) \boldsymbol{W} \quad \text{for } |\omega| \leq \frac{\pi}{M} \tag{3.49}$$

where $\boldsymbol{S}_u(M\omega)$ is the spectral density matrix of $\boldsymbol{u}(n)$ constructed from $u_i(n)$ in (2.52) by

$$\boldsymbol{u}(n) = (u_0(n), u_1(n), \ldots, u_{M-1}(n))^T.$$

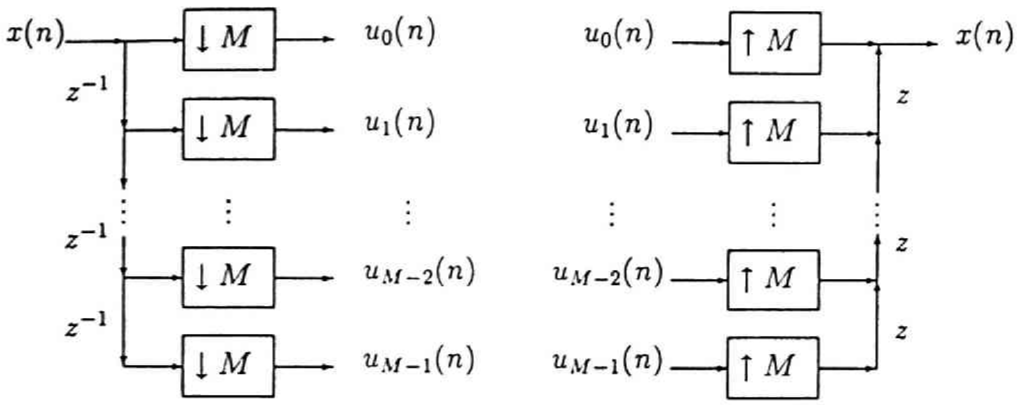

Figure 3.9: Figure of the relation $u_i(n) = x(Mn - i)$.

Consider Fig. 3.6 in place of Fig. 3.5 to easily analyze filter banks from the stochastic point of view. We use the above $\boldsymbol{u}(n)$ and $\boldsymbol{w}(n)$ and define an M-channel process $\boldsymbol{v}(n)$ as

$$\boldsymbol{v}(n) = (v_0(n), v_1(n), \ldots, v_{M-1}(n))^T. \tag{3.50}$$

First, we consider the analysis filter bank. $\boldsymbol{u}(n)$ can be expressed by

$$\boldsymbol{u}(n) = (x(Mn), x(Mn-1), \ldots, x(Mn-M+1))^T. \tag{3.51}$$

From the results of Section 2.1.3, since $x(n)$ is cyclostationary with period M, $\boldsymbol{u}(n)$ is an M-channel stationary process. From Gladyshev's relation (3.49), the relation between $\boldsymbol{F}_x(\omega)$ and $\boldsymbol{S}_u(\omega)$ is given by

$$\boldsymbol{S}_u(\omega) = \frac{1}{M}\Lambda(e^{j\frac{\omega}{M}})\boldsymbol{W}\boldsymbol{F}_x(\frac{\omega}{M})\boldsymbol{W}^{\dagger}\Lambda(e^{-j\frac{\omega}{M}}). \tag{3.52}$$

Since $\boldsymbol{v}(n)$ is filtered by $\boldsymbol{E}_h(z)$, it is also an M-channel stationary process whose spectral density matrix is given by

$$\begin{aligned}\boldsymbol{S}_v(\omega) &= \frac{1}{M}\boldsymbol{E}_h(e^{j\omega})\Lambda(e^{j\frac{\omega}{M}})\boldsymbol{W}\boldsymbol{F}_x(\frac{\omega}{M})\boldsymbol{W}^{\dagger}\Lambda(e^{-j\frac{\omega}{M}})\boldsymbol{E}_h^T(e^{-j\omega}) \\ &= \frac{1}{M}\boldsymbol{H}_{AC}^{\dagger}(e^{-j\frac{\omega}{M}})\boldsymbol{F}_x(\frac{\omega}{M})\boldsymbol{H}_{AC}(e^{-j\frac{\omega}{M}}).\end{aligned} \tag{3.53}$$

On the other hand, in the time domain, from (3.7),

$$v_i(n) = \sum_{l_1} h_i(l_1)x(Mn - l_1). \tag{3.54}$$

Thus we have

$$(\boldsymbol{R}_v(\tau))_{ik} = E[\, v_i(m+\tau)v_k(m)\,] \tag{3.55}$$

$$= \sum_{l_1,l_2} h_i(l_1)h_k(l_2)R_x(M\tau - l_1, -l_2). \tag{3.56}$$

Since $(\boldsymbol{S}_v(\omega))_{ik}$ is the discrete-time Fourier transform of the above equation, from (3.53) and (3.56), we have the following formula;

$$(\boldsymbol{S}_v(\omega))_{ik} = \frac{1}{2\pi}\sum_{\tau}\sum_{l_1,l_2} h_i(l_1)h_k(l_2)R_x(M\tau - l_1, -l_2)e^{-j\omega\tau} \tag{3.57}$$

$$= \sum_{m=0}^{M-1}\sum_{n=0}^{M-1} H_i^*(e^{-j(\omega+\frac{2\pi n}{M})})F_{m-n}(\frac{\omega+2\pi m}{M})H_i(e^{j(\omega+\frac{2\pi n}{M})}) \tag{3.58}$$

where $F_{(\cdot)}(\omega)$ denotes the spectral correlation density of $x(n)$.

Secondly, in the synthesis filter bank, since $v(n)$ is filtered by $\boldsymbol{E}_g^T(z)$ to produce the output $\boldsymbol{w}(n)$, $\boldsymbol{w}(n)$ is also an M-channel stationary with the spectral density matrix

$$\boldsymbol{S}_w(\omega) = \boldsymbol{E}_g^T(e^{j\omega})\boldsymbol{S}_v(\omega)\boldsymbol{E}_g(e^{-j\omega}). \tag{3.59}$$

The relation between $\widehat{x}(n)$ and $w_i(n)$ in Fig. 3.6 is given by

$$\widehat{x}(Mn+i) = w_i(n) \;\; \text{for } i = 0,\ldots,M-1. \tag{3.60}$$

This means that $\widehat{x}(n)$ is a cyclostationary process with period M. From Gladyshev's relation (3.48), its cyclic spectral density matrix is given by

$$\boldsymbol{F}(\omega) = \frac{1}{M}\boldsymbol{W}\boldsymbol{\Lambda}(e^{j\omega})\boldsymbol{S}_w(M\omega)\boldsymbol{\Lambda}(e^{-j\omega})\boldsymbol{W}^\dagger \;\; \text{for } |\omega| \le \frac{\pi}{M}. \tag{3.61}$$

Then substituting (3.59) into (3.61), we have

$$\boldsymbol{F}(\omega) = \frac{1}{M}\boldsymbol{G}_{AC}^*(e^{-j\omega})\boldsymbol{S}_v(M\omega)\boldsymbol{G}_{AC}^T(e^{-j\omega}) \tag{3.62}$$

where

$$\boldsymbol{G}_{AC}(z) = \boldsymbol{W}^\dagger\boldsymbol{\Lambda}(z)\boldsymbol{E}_g^T(z^M). \tag{3.63}$$

On the other hand, in the time domain, from (3.9) and from (3.60),

$$\widehat{x}(Mn+i) = \sum_{k_1} g_i(k_1)v_i(n-k_1). \tag{3.64}$$

Thus we get

$$
\begin{aligned}
R_{\hat{x}}(Mm+i, Mn+k) &= E[\,\hat{x}(Mm+i)\hat{x}(Mn+k)\,] \\
&= \sum_{k_1,k_2} g_i(k_1)g_k(k_2)(R_v(M(m-n)-k_1+k_2))_{ik}.
\end{aligned} \tag{3.65}
$$

Therefore, substituting (3.53) into (3.61), we finally get

$$
F(\omega) = \frac{1}{M^2}(H_{AC}(z)G_{AC}^T(z))^\dagger F_x(\omega) H_{AC}(z)G_{AC}^T(z)|_{z=e^{-j\omega}} \quad \text{for } |\omega| \le \frac{\pi}{M}. \tag{3.66}
$$

Similarly, in the time domain, substituting (3.56) into (3.65), we have

$$
\begin{aligned}
&R_{\hat{x}}(Mm+i, Mn+k) = \\
&\sum_{l_1,l_2,k_1,k_2} h_i(l_1)h_k(l_2)g_i(k_1)g_k(k_2)R_x(M^2(m-n)+M(k_2-k_1)-l_1,-l_2).
\end{aligned} \tag{3.67}
$$

(3.66) and (3.67) are the general input-output relations of filter banks in the transform domain and in the time domain, respectively.

It should be noted that if $x(n)$ is a stationary process with zero mean, the covariance $R_x(n)$, and the spectral density $S_x(\omega)$, then

$$
F_x(\omega) = \begin{pmatrix}
S_x(\omega) & 0 & \cdots & 0 \\
0 & S_x(\omega+\omega_0) & \ddots & \vdots \\
\vdots & \ddots & \ddots & 0 \\
0 & \cdots & 0 & S_x(\omega+(M-1)\omega_0)
\end{pmatrix} \tag{3.68}
$$

with $\omega_0 = 2\pi/M$. Now we show the following theorem.

Theorem 1

If a filter bank is alias free, then its output for any stationary input is stationary.

Proof

The output $\hat{x}(n)$ is stationary if its spectral correlation density matrix $F(\omega)$ is diagonal. From (3.66) it is sufficient to show that

$$
H_{AC}(z)G_{AC}^T(z) = W^\dagger \Lambda(z) E_h^T(z^M) E_g(z^M) \Lambda(z) W^\dagger \tag{3.69}
$$

is diagonal since $F_x(\omega)$ is diagonal.

Using Type 1 polyphase representation of $G(z)$, from (3.32) and (3.42), we have

$$\begin{aligned} \boldsymbol{E}_h^T(z)\boldsymbol{E}_g(z) &= \boldsymbol{E}_h^T(z)\boldsymbol{R}_g^T(z)J_M = \boldsymbol{P}^T(z)J_M \\ &= \begin{pmatrix} z^{-1}p_1(z) & z^{-1}p_2(z) & \cdots & z^{-1}p_{M-1}(z) & p_0(z) \\ z^{-1}p_2(z) & p_3(z) & \cdots & p_0(z) & p_1(z) \\ \vdots & \vdots & \iddots & \vdots & \vdots \\ p_0(z) & p_1(z) & \cdots & p_{M-2}(z) & p_{M-1}(z) \end{pmatrix}. \end{aligned} \tag{3.70}$$

By defining

$$\boldsymbol{Q}(z) = \boldsymbol{\Lambda}(z)\boldsymbol{E}_h^T(z^M)\boldsymbol{E}_g(z^M)\boldsymbol{\Lambda}(z), \tag{3.71}$$

it can be easily shown that $\boldsymbol{Q}(z)$ is a left circulant matrix

$$\boldsymbol{Q}(z) = \begin{pmatrix} q_1(z) & \cdots & q_{M-1}(z) & q_0(z) \\ q_2(z) & \cdots & q_0(z) & q_1(z) \\ \vdots & \iddots & \vdots & \vdots \\ q_0(z) & \cdots & q_{M-2}(z) & q_{M-1}(z) \end{pmatrix} \tag{3.72}$$

where

$$q_k(z) = z^{-M+1-k}p_k(z^M). \tag{3.73}$$

By using the property of the circulant matrix [7], this can be expressed as

$$\boldsymbol{Q}(z) = \sum_{k=0}^{M-1} q_k(z)J_M\Pi^k \tag{3.74}$$

with the permutation matrix

$$\Pi = \begin{pmatrix} 0 & I_{M-1} \\ 1 & 0 \end{pmatrix}. \tag{3.75}$$

Next we show that

$$\boldsymbol{W}^\dagger J_M\Pi^k\boldsymbol{W}^\dagger = M\boldsymbol{\Lambda}(W^{k-1}). \tag{3.76}$$

Postmultiplying the mth row of $\boldsymbol{W}^\dagger$ by $J_M\Pi^k$, we have

$$\begin{aligned} &(1, W^{-m}, W^{-2m}, \ldots, W^{-(M-1)m})J_M\Pi^k \\ =\ & (W^{-(M-1)m}, W^{-(M-2)m}, \ldots, W^{-km}, \ldots, 1)\Pi^k \\ =\ & (W^{-(k-1)m}, W^{-(k-2)m}, \ldots, 1, W^{-(M-1)m}, \ldots, W^{-km}). \end{aligned}$$

Then from (2.34), we have

$$\begin{aligned}(\boldsymbol{W}^{\dagger}J_M\Pi^k\boldsymbol{W}^{\dagger})_{mn} &= \sum_{l=0}^{M-1} W^{-(k-1-l)m}W^{-ln} = W^{-(k-1)m}\sum_{l=0}^{M-1} W^{l(m-n)} \\ &= MW^{-(k-1)m}\delta_{mn}\end{aligned}$$

where δ_{mn} denotes Kronecker's delta such that

$$\delta_{mn} = \begin{cases} 1 & m = n \\ 0 & m \neq n \end{cases}. \tag{3.77}$$

By using (3.37), (3.76) is obtained. Therefore from (3.71), (3.73), (3.74) and (3.76) we have

$$\begin{aligned}&\frac{1}{M}\boldsymbol{H}_{AC}(z)\boldsymbol{G}_{AC}^{T}(z) = \frac{1}{M}\boldsymbol{W}^{\dagger}\boldsymbol{Q}(z)\boldsymbol{W}^{\dagger} \\ &= \frac{1}{M}\sum_{k=0}^{M-1} q_k(z)\boldsymbol{W}^{\dagger}J_M\Pi^k\boldsymbol{W}^{\dagger} = z^{-M}\sum_{k=0}^{M-1} p_k(z^M)z^{-(k-1)}\boldsymbol{\Lambda}(W^{k-1}).\end{aligned} \tag{3.78}$$

Since $\boldsymbol{\Lambda}(W^{k-1})$ is diagonal, $\boldsymbol{H}_{AC}(z)\boldsymbol{G}_{AC}^{T}(z)$ is also diagonal. This shows that the output is stationary. □

Assume that the filter bank is alias free and let us denote the 0th diagonal element of (3.78) as

$$T(z) = z^{-M}\sum_{k=0}^{M-1} p_k(z^M)z^{-(k-1)}. \tag{3.79}$$

From (3.78), it can be easily shown that

$$\frac{1}{M}\boldsymbol{H}_{AC}(z)\boldsymbol{G}_{AC}^{T}(z) = \begin{pmatrix} T(z) & 0 & \cdots & 0 \\ 0 & T(zW) & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & T(zW^{M-1}) \end{pmatrix}. \tag{3.80}$$

It is noted that $T(z)$ in (3.79) is identical with $T(z)$ in (3.40).

From (2.41) and (3.66), for $i = 0, \ldots, M-1$, we have

$$F_0(\omega + \frac{2\pi i}{M}) = |T(e^{j(\omega+\frac{2\pi i}{M})})|^2 S_x(\omega + \frac{2\pi i}{M}) \tag{3.81}$$

for $|\omega| \leq \pi/M$, that is,

$$F_0(\omega) = |T(e^{j\omega})|^2 S_x(\omega) \quad \text{for } |\omega| \leq \pi. \tag{3.82}$$

In Section 3.1, we have seen the alias free filter bank and the PR filter bank from the deterministic point of view. From the above equation, we can see that the alias free filter bank characterized by (3.42) is also equivalent to a LTI system given by (3.79) from the stochastic point of view.

Particularly, if the filter bank has the PR property in the deterministic sense, from (3.46), we have

$$F_0(\omega) = c^2 S_x(\omega) \tag{3.83}$$

which shows that it also has the PR property in the stochastic sense. Intuitively, these may be natural results. But it is remarked that we have shown these explicitly and theoretically.

3.3 Perfect Reconstruction Filter Banks

In this section, M-band PR filter banks are briefly reviewed. In general, it is very difficult to parameterize the PR condition by filter coefficients except that $M = 2$. Then the tree structure filter bank with L-level in Fig. 3.10 is often used where $H_0^k(z)$, $H_1^k(z)$, $G_0^k(z)$ and $G_1^k(z)$ for each k satisfy the two-band PR condition. It is easily shown that the total filter bank has the perfect reconstruction. Therefore the M-band PR filter bank where $M = 2^L$ can be obtained. It is noted that by using the noble identities the filter bank in Fig. 3.10 can be rewritten as shown in Fig. 3.11.

As another special case, the binary structured PR filter bank in Fig. 3.12 is used for multiresolution analysis [23] [24] and to generate the wavelet mother function [8] [9].

The two-band PR filter banks are important in theory and in practice. Actually, there are a number of the two-band PR filter banks. Among them, we introduce the biorthogonal filter banks with the filters of same order. The conjugate quadrature filters (CQF) banks and the PR linear phase filter (LPF) banks are also treated. These can be obtained as special cases of the biorthogonal filter banks with additional constraints. The properties of the CQF banks are shown in [40] and those of the PR LPF banks are in [40] and [43].

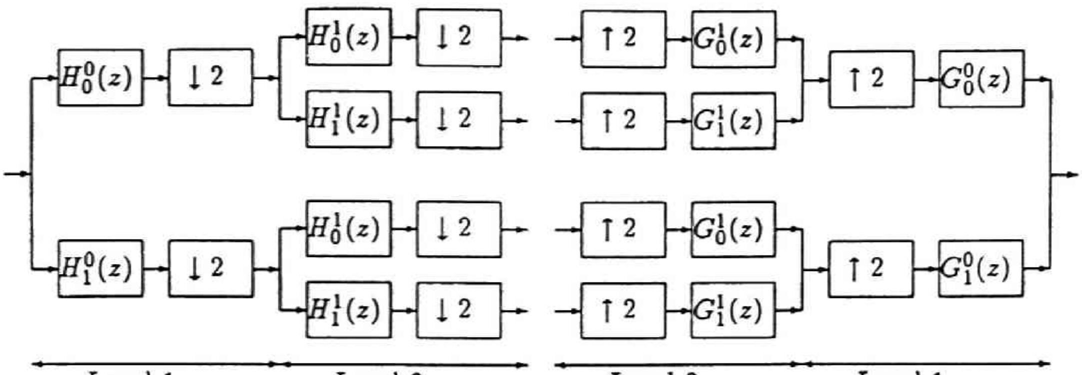

Figure 3.10: Tree structured filter bank with two level.

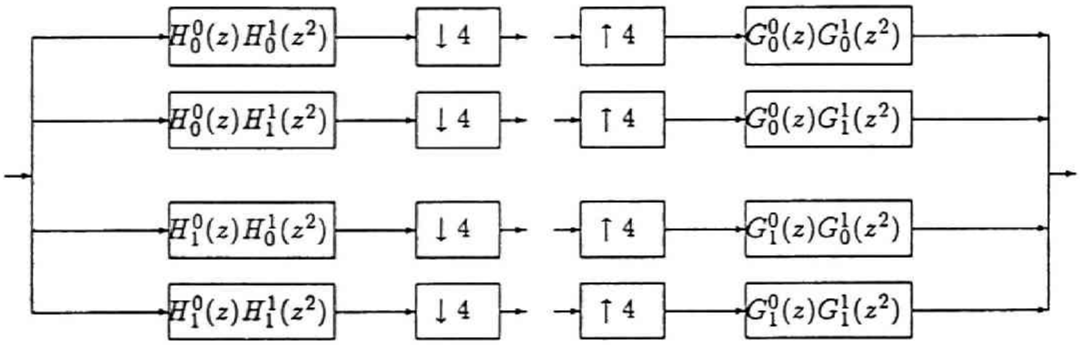

Figure 3.11: Equivalent filter bank to Fig. 3.10.

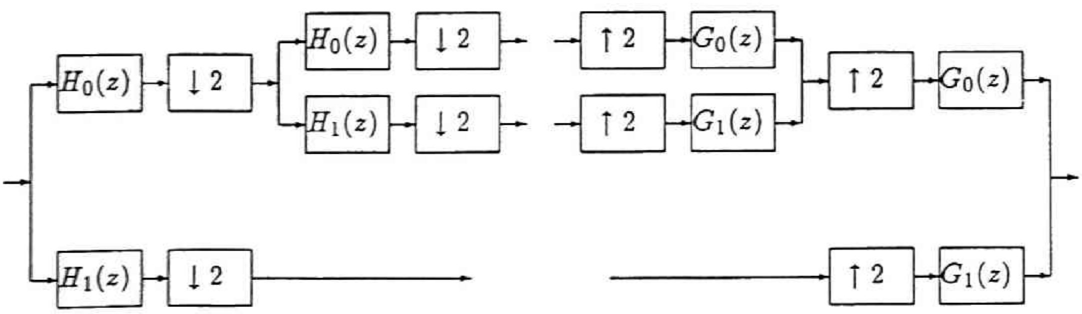

Figure 3.12: Binary structured filter bank with two level.

3.3.1 Biorthogonal filter banks

Let $H_0(z)$ and $H_1(z)$ are causal finite impulse response (FIR) filters with odd order N and put

$$G_0(z) = H_1(-z), \qquad G_1(z) = -H_0(-z). \tag{3.84}$$

Then, the PR condition (3.46) reduces to

$$T(z) = \frac{1}{2}(H_1(z)G_1(z) - H_1(-z)G_1(-z)) = cz^{-n_0} \quad n_0 : \text{odd integer}, c \neq 0. \tag{3.85}$$

This filter bank is called the biorthogonal filter bank [43]. We put $c = 1$ without loss of generality. For notational simplicity, we denote the corresponding coefficients $h_1(n)$ and $g_1(n)$ as h_n and g_n $(n = 0, \ldots, N)$, respectively, and define the following matrices and vectors;

$$G = \begin{pmatrix} g_0 & 0 & \cdots & & 0 \\ g_1 & g_0 & 0 & & \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ g_{N-1} & g_{N-2} & & g_0 & 0 \\ g_N & g_{N-1} & \ddots & g_1 & g_0 \\ 0 & g_N & & g_2 & g_1 \\ \vdots & 0 & \ddots & \vdots & \vdots \\ & & \ddots & g_N & g_{N-1} \\ 0 & & \cdots & 0 & g_N \end{pmatrix}, \quad G^- = \begin{pmatrix} g_0 & 0 & \cdots & & 0 \\ -g_1 & g_0 & 0 & & \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ g_{N-1} & -g_{N-2} & & g_0 & 0 \\ -g_N & g_{N-1} & \ddots & -g_1 & g_0 \\ 0 & -g_N & & g_2 & -g_1 \\ \vdots & 0 & \ddots & \vdots & \vdots \\ & & \ddots & -g_N & g_{N-1} \\ 0 & & \cdots & 0 & -g_N \end{pmatrix} \tag{3.86}$$

$$G_c = \begin{pmatrix} g_1 \ g_0 & 0 \ 0 & \cdots & & 0 \ 0 \\ g_3 \ g_2 & g_1 \ g_0 & 0 \ 0 & & \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ g_{N-2} \ g_{N-3} & & & g_1 \ g_0 & 0 \ 0 \\ g_N \ g_{N-1} & g_{N-2} \ g_{N-3} & \ddots & g_3 \ g_2 & g_1 \ g_0 \\ 0 \ 0 & g_N \ g_{N-1} & & & g_3 \ g_2 \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ & & 0 \ 0 & g_N \ g_{N-1} & g_{N-2} \ g_{N-3} \\ 0 \ 0 & & \cdots & 0 \ 0 & g_N \ g_{N-1} \end{pmatrix} \tag{3.87}$$

$$
\begin{aligned}
\boldsymbol{h} &= (h_0, h_1, \ldots, h_N)^T & (3.88)\\
\boldsymbol{h}^- &= (h_0, -h_1, \ldots, h_{N-1}, -h_N)^T & (3.89)\\
c(k) &= (0, \ldots, 0, \overset{\overset{k}{\downarrow}}{1}, 0, \ldots, 0)^T. & (3.90)
\end{aligned}
$$

It can be easily shown that the ith coefficient of $G_1(z)H_1(z)$ is the ith element of $\boldsymbol{Gh}$. Similarly, since the ith coefficient of $G_1(-z)H_1(-z)$ is the ith element of $\boldsymbol{G}^-\boldsymbol{h}^-$, we can rewrite the PR condition (3.85) as

$$
\boldsymbol{Gh} - \boldsymbol{G}^-\boldsymbol{h}^- = 2\boldsymbol{c}^T(n_0). \tag{3.91}
$$

After some computations, we have

$$
\begin{aligned}
(\boldsymbol{Gh})_{2k} &= (\boldsymbol{G}^-\boldsymbol{h}^-)_{2k} && \text{for } k = 0, \ldots, N\\
(\boldsymbol{Gh})_{2k+1} &= -(\boldsymbol{G}^-\boldsymbol{h}^-)_{2k+1} && \text{for } k = 0, \ldots, N-1.
\end{aligned} \tag{3.92}
$$

Thus (3.91) can be expressed by

$$
\boldsymbol{G_c h} = \boldsymbol{c}(L) \tag{3.93}
$$

where $L = (n_0 - 1)/2$. Here we only consider the case $n_0 = N$ and $h_0 \neq 0$.

In order to obtain the coefficients uniquely, we put $h_0 = 1$ and assume that the $N \times N$ submatrix $\boldsymbol{X}$ constructed from the first to the Nth column of $\boldsymbol{G_c}$ is nonsingular. By defining

$$
\begin{aligned}
\boldsymbol{a} &= (h_1, h_2, \ldots, h_N)^T\\
\boldsymbol{b} &= (g_1, g_3, \ldots, g_N, 0, \ldots, 0)^T,
\end{aligned}
$$

(3.93) can be rewritten by

$$
\begin{pmatrix} \boldsymbol{b} & \boldsymbol{X} \end{pmatrix} \begin{pmatrix} 1 \\ \boldsymbol{a} \end{pmatrix} = \boldsymbol{c}(L). \tag{3.94}
$$

Then, given g_n $(n = 0, \ldots, N)$, we can compute h_n by

$$
\boldsymbol{a} = \boldsymbol{X}^{-1}(\boldsymbol{c}(L) - \boldsymbol{b}). \tag{3.95}
$$

In this case, by considering the conditions on filter coefficients, it is shown that the total number of free coefficients of a biorthogonal filter bank is $N + 1$.

3.3.2 Conjugate quadrature filter banks

In addition the biorthogonal condition, put

$$H_1(z) = (-z)^{-N}\widetilde{H}_0(-z) \tag{3.96}$$

where

$$\widetilde{H}(z) = \sum_{n=0}^{N} h^{\dagger}(n)z^{n}, \tag{3.97}$$

or equivalently

$$h_1(n) = (-1)^{N} h_0^{*}(N-n). \tag{3.98}$$

In this case, the PR condition (3.85) reduces to

$$\widetilde{H}_0(z)H_0(z) + \widetilde{H}_0(-z)H_0(-z) = 2 \tag{3.99}$$

and

$$T(z) = z^{-N}. \tag{3.100}$$

The filters satisfying (3.96) and (3.99) are originally said to be the conjugate quadrature filters (CQF). By setting (3.84), the PR condition is satisfied as seen above. This filter bank is called the CQF bank that was derived before the biorthogonal filter bank. From the beginning of multirate signal processing, the CQF banks have been used in many fields and have been well studied. It is remarked that the CQF bank is also called the orthogonal filter bank or the PR QMF bank. But strictly speaking, the term "QMF" in signal processing is not appropriate since the QMF originally means that

$$H_1(z) = H_0(-z).$$

Now let us briefly mention the properties of the CQF banks. By noting that

$$\widetilde{H}(e^{j\omega}) = H^{*}(e^{j\omega}) \tag{3.101}$$

and from (3.99), it can be shown that

$$\frac{1}{2\pi}\int_{-\pi/2}^{\pi/2}\left(|H_0(e^{j\omega})|^2 + |H_0(-e^{j\omega})|^2\right)d\omega = 1. \tag{3.102}$$

So we have

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}|H_0(e^{j\omega})|^2 d\omega = 1. \tag{3.103}$$

From Parseval's relation, we have

$$\sum_{n=0}^{N} |h_0(n)|^2 = 1. \tag{3.104}$$

From (3.84), (3.96) and (3.99), it can be also shown that

$$\begin{aligned} \widetilde{H}_i(z)H_i(z) + \widetilde{H}_i(-z)H_i(-z) &= 2 & (3.105) \\ \tilde{G}_i(z)G_i(z) + \tilde{G}_i(-z)G_i(-z) &= 2 & (3.106) \end{aligned}$$

for $i = 0, 1$. Then, by the same derivation of (3.104), we have

$$\begin{aligned} \sum_{n=0}^{N} |h_i(n)|^2 &= 1 & (3.107) \\ \sum_{n=0}^{N} |g_i(n)|^2 &= 1 & (3.108) \end{aligned}$$

for $i = 0, 1$.

It can be easily shown that the AC matrix of a CQF bank satisfies

$$\widetilde{\boldsymbol{H}}_{AC}(z)\boldsymbol{H}_{AC}(z) = 2I. \tag{3.109}$$

The above equation is called the paraunitary condition. Conversely if the AC matrix of a two-band filter bank satisfies the paraunitary condition, then the filter bank is a CQF bank [40]. It is also shown in [40] that a CQF bank can be parameterized by $(N + 1)/2$ free parameters.

Lastly, we show that the averaged variances of subband signals of the CQF banks are equal to the variance of the input signal, that is,

$$\frac{1}{2}(\sigma_{v_0}^2 + \sigma_{v_1}^2) = \sigma_x^2 \tag{3.110}$$

where $\sigma_{v_i}^2$ and σ_x^2 denote the variances of the ith subband signal and the input signal, respectively.

From (3.56), it can be shown that

$$\sigma_{v_i}^2 = \sum_{l=0}^{M-1} \sum_{m=0}^{M-1} h_i(l) R_x(m-l) h_i(m). \tag{3.111}$$

for $i = 0, 1$ where and $R_x(n)$ denotes the auto-correlation of the input signal. Then, from (3.99) and Parseval's relation, we have

$$
\begin{aligned}
&\frac{1}{2}(\sigma_{v_0}^2 + \sigma_{v_1}^2) \\
&= \frac{1}{2}\sum_{l=0}^{M-1}\sum_{m=0}^{M-1}(h_0(l)R_x(m-l)h_0(m) + h_1(l)R_x(m-l)h_1(m).) \\
&= \frac{1}{2}\int_{-\pi}^{\pi}\left(|H_0(e^{j\omega})|^2 + |H_1(e^{j\omega})|^2\right)S_x(\omega)d\omega \\
&= \int_{-\pi}^{\pi} S_x(\omega)d\omega = \sigma_x^2
\end{aligned}
$$

where $S_x(\omega)$ denotes the spectral density of the input signal. This fact is used in Section 3.4.1.

3.3.3 Perfect reconstruction linear phase filter banks

In some applications it is desirable to have a filter bank whose analysis filters $H_0(z)$ and $H_1(z)$ have linear phase. It is known that the FIR filter with real coefficients has linear phase if $h(n)$ is symmetric or antisymmetric, that is,

$$h(n) = h(N-n)$$

or

$$h(n) = -h(N-n).$$

But we can not use the CQF banks because they do not have linear phase except for some trivial cases [40]. It is also shown in [43] that if the orders of filters are same and odd then, in addition to the biorthogonal condition, the low pass filter $H_0(z)$ and the high pass filter $H_1(z)$ must satisfy

$$h_0(n) = h_0(N-n), \tag{3.112}$$

$$h_1(n) = -h_1(N-n) \tag{3.113}$$

for $n = 0, \ldots, N$.

Then using the polyphase representation,

$$\begin{pmatrix} H_0(z) \\ H_1(z) \end{pmatrix} = \begin{pmatrix} E_{00}(z^2) & E_{01}(z^2) \\ E_{10}(z^2) & E_{11}(z^2) \end{pmatrix} \begin{pmatrix} 1 \\ z^{-1} \end{pmatrix} \tag{3.114}$$

and (3.112), it can be easily shown that

$$E_{01}(z^2) = \sum_{l=0}^{(N-1)/2} h_0(2l)z^{2l}z^{-N+1} = z^{-N+1}E_{00}(z^{-2}) \tag{3.115}$$

Similarly, we also have

$$E_{11}(z^2) = -z^{-N+1}E_{10}(z^{-2}). \tag{3.116}$$

For notational simplicity, we rewrite $D(z) = E_{00}(z)$ and $F(z) = E_{10}(z)$ with the corresponding filter coefficients d_n and f_n $(n = 0, \ldots, L)$, respectively where

$$L = \frac{N-1}{2}. \tag{3.117}$$

If we put $G_0(z)$ and $G_1(z)$ as in (3.84), then from (3.84) and (3.114) we have

$$\begin{pmatrix} H_0(z) \\ H_1(z) \end{pmatrix} = \begin{pmatrix} D(z^2) & D(z^{-2}) \\ F(z^2) & -F(z^{-2}) \end{pmatrix} \begin{pmatrix} 1 \\ z^{-N} \end{pmatrix} \tag{3.118}$$

$$\begin{pmatrix} G_0(z) \\ G_1(z) \end{pmatrix} = \begin{pmatrix} F(z^2) & F(z^{-2}) \\ -D(z^2) & D(z^{-2}) \end{pmatrix} \begin{pmatrix} 1 \\ z^{-N} \end{pmatrix}. \tag{3.119}$$

Substituting $G_1(z)$ and $H_1(z)$ into $T(z)$ yields

$$T(z) = z^{-N}(D(z^2)F(z^{-2}) + D(z^{-2})F(z^2)) = z^{-n_0}. \tag{3.120}$$

Thus if we can find two filters $D(z)$ and $F(z)$ satisfying

$$D(z)F(z^{-1}) + D(z^{-1})F(z) = 1, \tag{3.121}$$

then

$$T(z) = z^{-N} = z^{-n_0}. \tag{3.122}$$

This shows that the PR condition is satisfied. It is shown in [43] that n_0 must be equal to N. Therefore for the perfect reconstruction linear phase filter (PR LPF) banks, the PR condition is (3.121).

By putting $f_n = 0$ $(n < 0, n > L)$, (3.121) can be written in the time domain as

$$\sum_{k=-L}^{L} \sum_{i=0}^{L} d_i(f_{i-k} + f_{i+k})z^{-k} = 1. \tag{3.123}$$

Since the coefficient of z^l is equal to that of z^{-l} ($l = 0, \ldots, L$) in (3.123), we can rewrite (3.123) as

$$\left(\begin{pmatrix} f_0 & f_1 & \cdots & f_L \\ 0 & f_0 & \cdots & f_L \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & f_0 \end{pmatrix} + \begin{pmatrix} f_0 & \cdots & f_{L-1} & f_L \\ f_1 & \cdots & f_L & 0 \\ \vdots & .\cdot^{\cdot} & .\cdot^{\cdot} & \vdots \\ f_L & 0 & \cdots & 0 \end{pmatrix} \right) \begin{pmatrix} d_0 \\ d_1 \\ \vdots \\ d_L \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \tag{3.124}$$

When the matrix constructed from f_n in (3.124) is nonsingular, for given f_n, we can compute d_n uniquely. In this case, by considering the conditions on filter coefficients, it is shown that the total number of free coefficients of a PR LPF bank is $(N-1)/2$

3.4 Optimization of Filter Banks

In using two-band PR filter banks, in theory, it is desirable that the low pass and the high pass filter are ideal filters as shown in Fig. 3.13. But in practice, these filter can not be used since they needs infinite filter coefficients. Fortunately, as discussed in Section 3.3, PR filter banks still have freedoms of their coefficients so that we can construct the "good" PR filter bank according to the object of using the PR filter bank.

The main purpose of using filter banks in signal processing are to compress the input signal into some subband signals in video signal processing, to decompose the input signal into some subband signals whose correlation are small in subband signal processing, and to minimize the quantizing effect in subband coding.

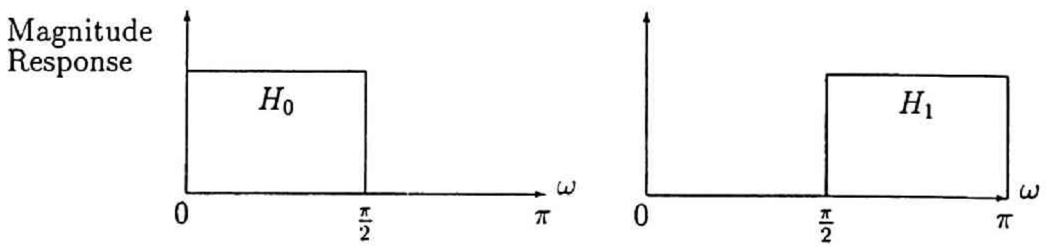

Figure 3.13: Frequency responses of the ideal 2-band filter bank.

In this section, we optimize a filter bank with a stationary input signal $x(n)$ by minimizing the averaged variance of the reconstruction error under the PR condition when some subband signals are dropped. We assume that high pass bands from K to M are dropped and denote the output of the filter bank as $y(n)$.

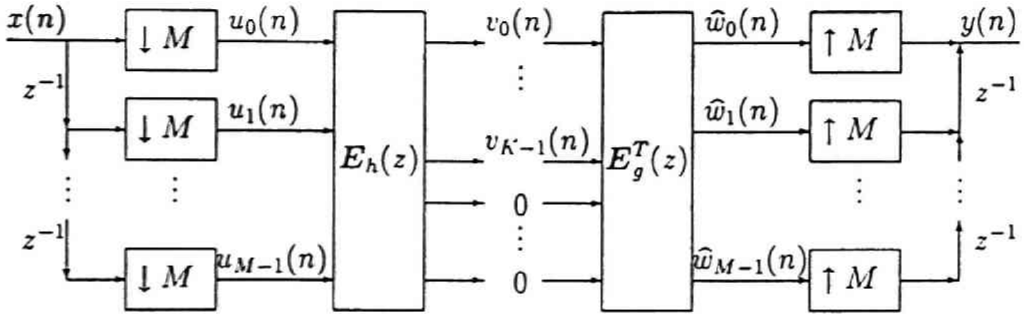

Figure 3.14: M-band filter bank when K to $M-1$ band signals are dropped

See Fig. 3.14 and define an M-channel process $\widehat{w}(n)$ as

$$\widehat{w}(n) = (\widehat{w}_0(n), \widehat{w}_1(n), \ldots, \widehat{w}_{M-1}(n))^T. \tag{3.125}$$

The difference between $w(n)$ and $\widehat{w}(n)$, denoted by $e(n)$, is given by

$$\begin{aligned} e(n) &= w(n) - \widehat{w}(n) = E_g^T(q)v(n) - E_g^T(q)\begin{pmatrix} I_{M-K} & 0 \\ 0 & 0 \end{pmatrix} v(n) \\ &= E_g^T(q)I'v(n) \end{aligned} \tag{3.126}$$

where

$$I' = \begin{pmatrix} 0 & 0 \\ 0 & I_K \end{pmatrix} \tag{3.127}$$

and q^{-1} is the delay operator $q^{-1}x(n) = x(n-1)$.

Since $v(n)$ is an M-channel stationary process, so is $e(n)$. Then as in (3.32)

$$y(Mn+i) = \widehat{w}_i(n) \tag{3.128}$$

so that, by defining the reconstruction error as $e(n) = \tilde{x}(n) - y(n)$, we have

$$(e(n))_i = \tilde{x}(Mn+i) - y(Mn+i) = e(Mn+i), \tag{3.129}$$

that is,

$$e(n) = (e(Mn), e(Mn+1), \ldots, e(Mn+M-1))^T. \tag{3.130}$$

Since the spectral density matrix of $e(n)$ is given by

$$S_e(\omega) = E_g(e^{j\omega}) I' S_v(\omega) I' E_g^T(e^{-j\omega}). \tag{3.131}$$

and from (3.26) and Gladyshev's relation (3.48), by using $H_{AC}(z)$ and $G_{AC}(z)$, the spectral correlation density matrix of $e(n)$ is given by

$$F_e(\omega) = \frac{1}{M^2}(H_{AC}(z) I' G_{AC}^T(z))^{\dagger} S_x(M\omega) H_{AC}(z) I' G_{AC}^T(z)|_{e^{-j\omega}} \quad \text{for } |\omega| \le \pi/M. \tag{3.132}$$

From (2.32), the averaged variance of the reconstruction error is given by

$$\sigma_e^2 = \int_{-\pi/M}^{\pi/M} \mathrm{tr} F_e(\omega) d\omega. \tag{3.133}$$

When $M = 2$ and the high pass band signal is dropped, from

$$\begin{aligned} H_{AC}(z) I' G_{AC}^T(z) &= \begin{pmatrix} H_0(z) & H_1(z) \\ H_0(-z) & H_1(-z) \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} G_0(z) & G_0(-z) \\ G_1(z) & G_1(-z) \end{pmatrix} \\ &= \begin{pmatrix} H_1(z) \\ H_1(-z) \end{pmatrix} \begin{pmatrix} G_1(z) & G_1(-z) \end{pmatrix}, \end{aligned}$$

(3.133) reduces to

$$\begin{aligned} \sigma_e^2 &= \frac{1}{4}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \mathrm{tr}\{ \begin{pmatrix} G_1(e^{j\omega}) \\ G_1(-e^{j\omega}) \end{pmatrix} \begin{pmatrix} H_1(e^{j\omega}) & H_1(-e^{j\omega}) \end{pmatrix} \cdot \\ &\quad \begin{pmatrix} S_x(\omega) & 0 \\ 0 & S_x(\omega+\pi) \end{pmatrix} \begin{pmatrix} H_1(e^{-j\omega}) \\ H_1(-e^{-j\omega}) \end{pmatrix} \begin{pmatrix} G_1(e^{-j\omega}) & G_1(-e^{-j\omega}) \end{pmatrix} \} d\omega \\ &= \frac{1}{4}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(|G_1(e^{j\omega})|^2 + |G_1(e^{j(\omega+\pi)})|^2 \right) |H_1(e^{j\omega})|^2 S_x(\omega) d\omega \\ &\qquad + \frac{1}{4}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(|G_1(e^{j\omega})|^2 + |G_1(e^{j(\omega+\pi)})|^2 \right) |H_1(e^{j(\omega+\pi)})|^2 S_x(\omega+\pi) d\omega \\ &= \frac{1}{4}\int_0^{2\pi} \left(|G_1(e^{j\omega})|^2 + |G_1(e^{j(\omega+\pi)})|^2 \right) |H_1(e^{j\omega})|^2 S_x(\omega) d\omega, \end{aligned} \tag{3.134}$$

since $|G_1(e^{j\omega})|^2 + |G_1(e^{j(\omega+\pi)})|^2$, $|H_1(e^{j\omega})|^2$ and $S_x(\omega)$ are periodic functions of ω with period 2π. Then the problem becomes to design one pair of filters among $H_0(z)$,

$H_1(z)$, $G_0(z)$ and $G_1(z)$ that minimizes (3.134) under the PR condition (3.85). Here the pair $(H_1(z), G_1(z))$ is chosen and let the order N be odd.

The integrand of (3.134) can be considered as the sum of the spectral densities of two signals $G_1(q)H_1(q)x(n)$ and $G_1(-q)H_1(q)x(n)$. Then the ith coefficients of $G_1(z)H_1(z)$ and $G_1(-z)H_1(z)$ are given by the ith elements of $\boldsymbol{G}\boldsymbol{h}$ and $\boldsymbol{G}^{-}\boldsymbol{h}$, respectively.

In the time domain, the criterion function (3.134) can be written as

$$\sigma_e^2 = \frac{1}{4}\left((\boldsymbol{G}\boldsymbol{h})^T \boldsymbol{R}_{2N+1}\boldsymbol{G}\boldsymbol{h} + (\boldsymbol{G}^{-}\boldsymbol{h})^T \boldsymbol{R}_{2N+1}\boldsymbol{G}^{-}\boldsymbol{h}\right) \tag{3.135}$$

with the $l \times l$ matrix $\boldsymbol{R}_l$;

$$(\boldsymbol{R}_l)_{ik} = R_x(i-k) \ \ 0 \leq i, k < l. \tag{3.136}$$

It is noted that this criterion is easily computed from the filter coefficients and the auto-correlation of the input.

For biorthogonal filter banks, the optimization problem becomes to minimize (3.135) subject to (3.93). Substituting (3.95) into (3.135) and minimizing (3.135) with respect to g_n $(n = 0, \ldots, N)$, we can obtain the optimal biorthogonal filer banks.

For CQF banks, from (3.106), the criterion (3.134) reduces to

$$\frac{1}{4}\int_0^{2\pi}\left(|G_1(e^{j\omega})|^2 + |G_1(e^{j(\omega+\pi)})|^2\right)|H_1(e^{j\omega})|^2 S_x(\omega)d\omega = \frac{1}{2}\boldsymbol{h}^T\boldsymbol{R}_{N+1}\boldsymbol{h}. \tag{3.137}$$

Then under the PR condition (3.99), by minimizing (3.137), the optimal CQF banks can be obtained. This optimization problem is originally considered in [41].

For PR LPF banks, from (3.119), we have

$$\begin{aligned} |G_1(e^{j\omega})|^2 &= |-D(e^{j2\omega}) + D(e^{-j2\omega})e^{-jN\omega}|^2 & (3.138)\\ &= 2|D(e^{j2\omega})|^2 - 2Re(D(e^{j2\omega})e^{jN\omega}), & (3.139)\end{aligned}$$

and

$$|G_1(e^{j(\omega+\pi)}|^2 = 2|D(e^{j2\omega})|^2 + 2Re(D(e^{j2\omega})e^{jN\omega}). \tag{3.140}$$

Then, the criterion (3.134) reduces to

$$\frac{1}{4}\int_0^{2\pi}\left(|G_1(e^{j\omega})|^2 + |G_1(e^{j(\omega+\pi)})|^2\right)S_x(\omega)d\omega = \int_0^{2\pi}|D(e^{j2\omega})H_1(e^{j\omega})|^2 S_x(\omega)d\omega. \tag{3.141}$$

Therefore, under the PR condition (3.121), by minimizing (3.141), the optimal PR LPF banks can be obtained.

3.4.1 Other criteria

There are some performance measures of PR filter banks according to the object of using the PR filter bank.

A measure of the efficiency to compress the signal is given by

$$\eta = \frac{\sigma_x^2 - \sigma_e^2}{\sigma_x^2}. \tag{3.142}$$

where σ_x^2 is the variance of the input $x(n)$ and σ_e^2 is the averaged variance of the reconstruction error when some subband signals are dropped. It is clear from (3.142) that minimizing σ_e^2 is equivalent to maximizing η. Then, our criterion for the optimal PR filter bank that minimizes the averaged variance of the reconstruction error where subband signals from K to M are dropped is equivalent to the criterion for the optimal PR filter bank that maximizes the efficiency η.

A measure of the efficiency of the decorrelation is given by

$$\beta = (\sum_{\tau} \sum_{i,k=0,i\neq k}^{M-1} |(R_v(\tau))_{ik}|^l)^{\frac{1}{l}} / \sigma_x^2 \tag{3.143}$$

where

$$(R_v(\tau))_{ik} = E[\, v_i(m+\tau) v_k(m)\,] \tag{3.144}$$

and l is a certain positive integer. From (3.56), we have

$$(R_v(\tau))_{ik} = \sum_{l=0}^{M-1} \sum_{m=0}^{M-1} h_i(l) R_x(m - l + M\tau) h_k(m). \tag{3.145}$$

The CQF filter banks that minimize the above measure are considered in [25]. Instead of the CQF condition (3.96), if the LPF condition (3.112) is used, then the optimal decorrelation PR LPF banks are also obtained.

Usually after being quantized the subband signals are transmitted to the synthesis bank. In the above two cases, the effect of the quantizing is ignored. Here we consider this and let the output of the kth quantizer be $\widehat{v}_k(n)$ and the corresponding quantizing error be

$$n_k(n) = v_k(n) - \widehat{v}_k(n). \tag{3.146}$$

When r_k bits are allocated to the quantizer Q_k, the pdf-optimized quantizer minimizing separately each mean square error [21] is used where

$$E[\ n_k(n)\ ] \ = \ 0 \tag{3.147}$$

$$E[\ n_k(n)\widehat{v}_k(n)\ ] \ = \ 0. \tag{3.148}$$

It is also shown in [21] that, under the assumption that each $v_k(n)$ is Gaussian, its variance $\sigma^2_{n_k}$ can be approximated by

$$\sigma^2_{n_k} = \gamma 2^{-2r_k}\sigma^2_{v_k}. \tag{3.149}$$

Define an M-channel process $\widehat{v}(n)$ as

$$\widehat{v}(n) = v(n) + n(n) \tag{3.150}$$

where

$$\widehat{v}(n) \ = \ (\widehat{v}_0(n), \ldots, \widehat{v}_{M-1}(n))^T \tag{3.151}$$

$$n(n) \ = \ (n_0(n), \ldots, n_{M-1}(n))^T. \tag{3.152}$$

For simplicity of analysis, assume $n_i(n)$ for $i = 0, M-1$ are mutually uncorrelated and

$$E[\ \widehat{v}(n+\tau)n^T(n)\ ] = 0 \text{ for any } n \text{ and } \tau. \tag{3.153}$$

From this assumption, we have

$$R_{\hat{v}}(\tau) = R_v(\tau) + R_n(\tau) \tag{3.154}$$

where $R_{\hat{v}}(\tau)$ and $R_n(\tau)$ denote the covariance matrix of $\widehat{v}(n)$ and that of $n(n)$, respectively. Also from the assumption, $R_n(\tau)$ is given by

$$R_n(\tau) = diag(\sigma^2_{n_0}, \ldots, \sigma^2_{n_{M-1}})\delta_{\tau,0}. \tag{3.155}$$

Since the noise $n(n)$ is filtered by the synthesis filter bank, by the same derivation in Section 3.2, the averaged mean square reconstruction error is given by

$$\sigma^2 = \frac{1}{M}\int_{-\pi/M}^{\pi/M} \mathrm{tr}(G^T_{AC}(e^{-j\omega})^\dagger S_n(M\omega) G^T_{AC} e^{-j\omega}))d\omega \tag{3.156}$$

where $S_n(M\omega)$ is the diagonal spectral density matrix of $n(n)$ whose ith diagonal element is $\sigma_{n_i}^2/2\pi$. Then (3.156) reduces to

$$\begin{aligned}
& \frac{1}{M}\int_{\pi/M}^{\pi/M}\sum_{k=0}^{M-1}\sum_{i=0}^{M-1}\frac{1}{2\pi}\sigma_{n_i}^2 G_i(e^{j\omega+\frac{2\pi k}{M}})G_i(e^{-(j\omega+\frac{2\pi k}{M})})d\omega \\
= & \frac{1}{M}\sum_{i=0}^{M-1}\sigma_{n_i}^2\sum_{k=0}^{M-1}\frac{1}{2\pi}\int_{\pi/M}^{\pi/M} G_i(e^{j\omega+\frac{2\pi k}{M}})G_i(e^{-(j\omega+\frac{2\pi k}{M})})d\omega \\
= & \frac{1}{M}\sum_{i=0}^{M-1}\sigma_{n_i}^2\frac{1}{2\pi}\int_{0}^{2\pi} G_i(e^{j\omega})G_i(e^{-j\omega})d\omega \\
= & \frac{1}{M}\sum_{i=0}^{M-1}\sigma_{n_i}^2(\sum_{n=0}^{N}|g_i(n)|^2)
\end{aligned} \tag{3.157}$$

where $g_i(n)$ are the ith coefficients of $G_i(z)$.

Let total bits r for quantization be constant, that is,

$$Mr = \sum_{k=0}^{M-1} r_k. \tag{3.158}$$

Under this condition, we minimize the reconstruction error (3.157). By using Lagrange multipliers, r_k and σ^2 are given by

$$r_k = r + \frac{1}{2}\log\frac{\sigma_{v_k}^2\sum_{n=0}^{N}|g_k(n)|^2}{(\Pi_{i=0}^{M-1}\sigma_{v_i}^2\sum_{n=0}^{N}|g_i(n)|^2)^{\frac{1}{M}}} \tag{3.159}$$

$$\sigma_{n_i}^2 = \gamma 2^{-2r}\frac{(\Pi_{i=0}^{M-1}\sigma_{v_i}^2\sum_{n=0}^{N}|g_i(n)|^2)^{\frac{1}{M}}}{\sum_{n=0}^{N}|g_i(n)|^2} \tag{3.160}$$

$$\sigma^2 = \gamma 2^{-2r}(\Pi_{i=0}^{M-1}\sigma_{v_i}^2\sum_{n=0}^{N}|g_i(n)|^2)^{\frac{1}{M}}, \tag{3.161}$$

respectively. The coding gain defined by

$$G_{SBC} = \frac{\sigma_x^2}{(\Pi_{i=0}^{M-1}\sigma_{v_i}^2\sum_{n=0}^{N}|g_i(n)|^2)^{\frac{1}{M}}} \tag{3.162}$$

is usually used as a measure of the efficiency of reducing the quantizing effect [38] [22]. It is also noted that, as in other criteria, the two-band optimal coding gain PR LPF bank can be obtained.

The optimal PR filter banks in terms of this coding gain are treated in [3] and [41]. For CQF banks, from (3.99) and (3.108), it can be easily shown that minimizing the criterion (3.137) is equivalent to minimizing σ^2 in (3.161). So the two-band optimal compression CQF bank is also the two-band optimal coding gain CQF bank. In some literature, since the above result has been obtained, the optimal coding gain filter bank is mistakenly regarded as the optimal compression filter bank. By comparing (3.134) with (3.162), it is obvious that this is not necessary true in the biorthogonal bank. So if the optimal compression biorthogonal filter banks are needed, our criterion should be used.

3.4.2 Numerical results

We optimize PR filter banks in terms of our criterion, using the Newton-Raphson method for optimization. It is difficult to obtain a global solution of a non-linear optimization problem with many variables. Since the biorthogonal filter bank treated in the Section 3.3.1 has twice as many coefficients as the special filter banks, we use the coefficients of the optimal CQF banks as the initial values for optimization of the biorthogonal filter bank.

The input signal with zero mean and the covariance $R_x(n) = 0.9^{-|n|}$ is used. Fig. 3.15, Fig. 3.16 and Fig. 3.17, respectively show their frequency responses of $H_0(z)$ and that of $H_1(z)$ with order $N = 7$. For the CQF bank, we can see that $H_1(z)$ is comparatively good high pass filter. For the PR LPF bank , $H_1(z)$ is not very good high pass filter. But this is a natural result since $D(z^2)H_1(z)$ in (3.141) in place of $H_1(z)$ has to be good high pass filter, which is clarified as shown in Fig. 3.18.

Table 3.1 shows the averaged variance of the reconstruction error σ_e^2 for each N. It should be noted that biorthogonal filter banks, CQF banks and PR LPF banks have N, $(N+1)/2$ and $(N-1)/2$ free parameters, respectively. We can see the averaged variance of the error of the CQF bank is smaller than that of the PR LPF bank for each N.

Table 3.2 shows three measures of the PR filter banks of Fig. 3.15, Fig. 3.16 and Fig. 3.17. Since we optimize PR filter banks in terms of the compression, it is natural that the biorthogonal filter bank is best in η. But it is instructive that the PR LPF bank is best in terms of decorrelation.

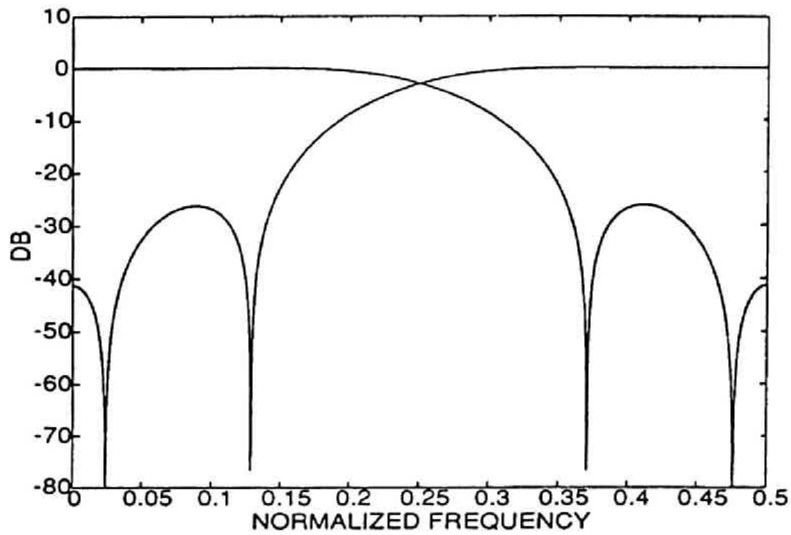

Figure 3.15: Frequency responses $H_0(e^{j\omega})$ and $H_1(e^{j\omega})$ of an 8-tap optimal CQF bank ($N = 7$).

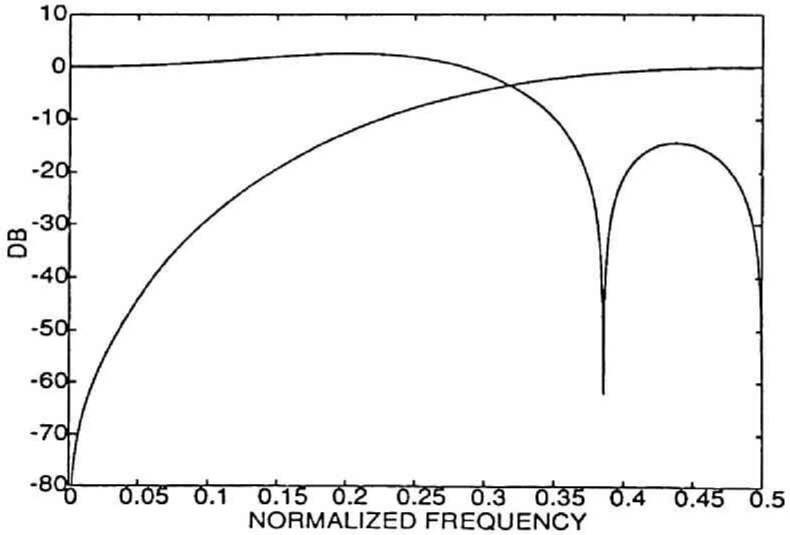

Figure 3.16: Frequency responses $H_0(e^{j\omega})$ and $H_1(e^{j\omega})$ of an 8-tap optimal PR LPF bank ($N = 7$).

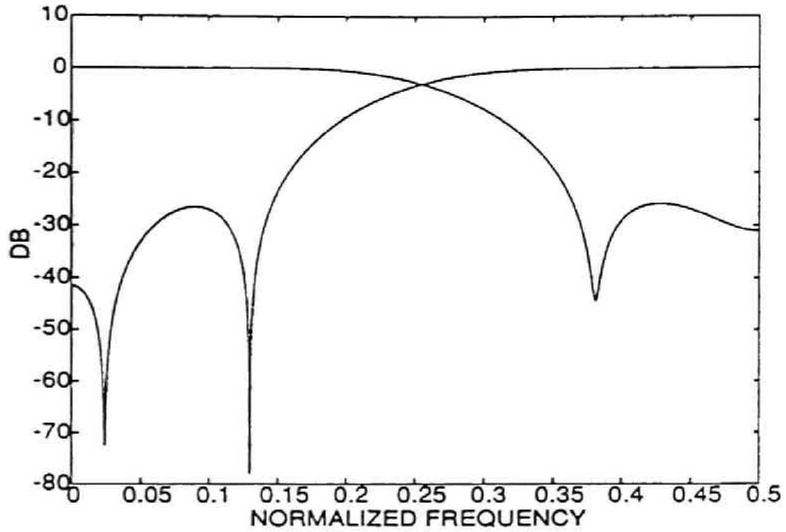

Figure 3.17: Frequency responses $H_0(e^{j\omega})$ and $H_1(e^{j\omega})$ of an 8-tap optimal PR bank ($N = 7$).

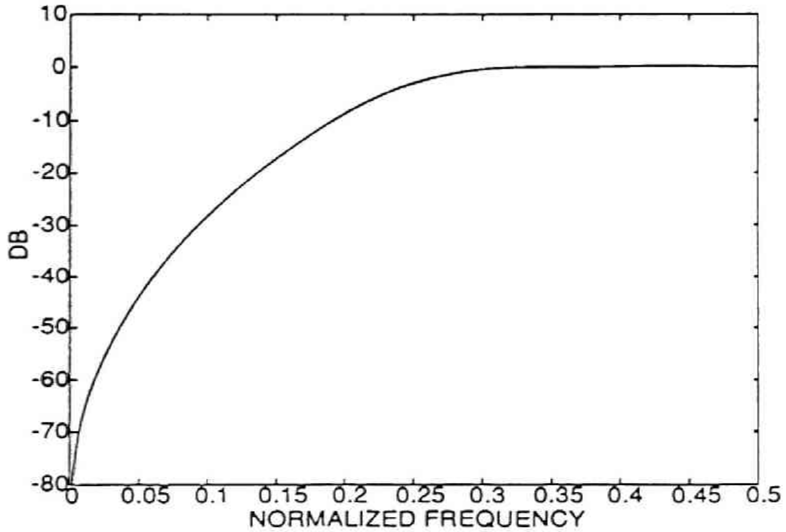

Figure 3.18: Frequency response of $D(e^{j2\omega})H_1(e^{j\omega})$ ($N = 7$).

N	CQF	PR LPF	Biorthogonal
1	5.000		
3	3.8916	4.1433	3.8846
5	3.6127	3.7590	3.6045
7	3.5031	3.5815	3.4997

Table 3.1: The average variances of the reconstruction errors ($\times 10^{-2}$).

$N = 7$	CQF	PR LPF	Biorthogonal
$\eta(\%)$	96.497	96.419	96.501
β (l=2)	0.0389	0.0173	0.0373
G_{SBC} (dB)	4.34	4.82	4.41

Table 3.2: Properties of the obtained optimal PR filter banks.

Chapter 4

NEW RESULTS ON PERIODIC AR PROCESSES

This chapter deals with periodic ARMA processes. In Section 4.1, the linear periodically time-varying systems are reviewed by using the cyclostationary analysis and the multirate system theory. In Section 4.2, periodic AR processes as special cases of periodic ARMA processes are further studied. New results are shown about the existence and the construction method of backward periodic AR processes from the circular Levinson algorithm. The statistical properties of the estimated coefficients of backward periodic AR processes based on a sample of finite size are also derived.

4.1 Linear Periodically Time-Varying Systems

$$\epsilon(t) \longrightarrow \boxed{H_t(z)} \longrightarrow y(t)$$

Figure 4.1: The general input-output relation.

In this chapter, we only treat real processes and filters with real coefficients.

Now periodic autoregressive moving average (ARMA) systems are considered from the theory of multirate systems. The discussion on periodic ARMA processes is based on Sakai [35].

Consider the following general causal linear system shown in Fig. 4.1;

$$y(t) = \sum_{m=0}^{\infty} h_t(t-m)\epsilon(m) \tag{4.1}$$

where the impulse response are time-varying. Assume that each transfer function

$$H_t(z) = \sum_{n=0}^{\infty} h_t(n) z^{-n} \tag{4.2}$$

is stable.

Among time-varying systems, the system whose impulse response satisfies

$$h_t(n) = h_{t+M}(n) \tag{4.3}$$

is called the linear periodically time-varying (LPTV) system. Each transfer function also satisfies

$$H_t(z) = H_{t+M}(z), \tag{4.4}$$

which is called the LPTV transfer function with period M.

It should be noted that, since the LTI transfer function satisfies

$$H_t(z) = H(z) \text{ for all } t, \tag{4.5}$$

the LPTV system can be considered to be an extension of the LTI system.

Setting $t = k + Mn$ in (4.1) for $k = 0, \ldots, M-1$, we have

$$y(k + Mn) = \sum_{m=0}^{\infty} h_k(k + Mn - m)\epsilon(m). \tag{4.6}$$

This shows that filtering $\epsilon(t)$ by the filter $H_k(z)$ produces the output $y(t)$ at the time t where the reminder of t divided by M is k. So by using the notations of multirate

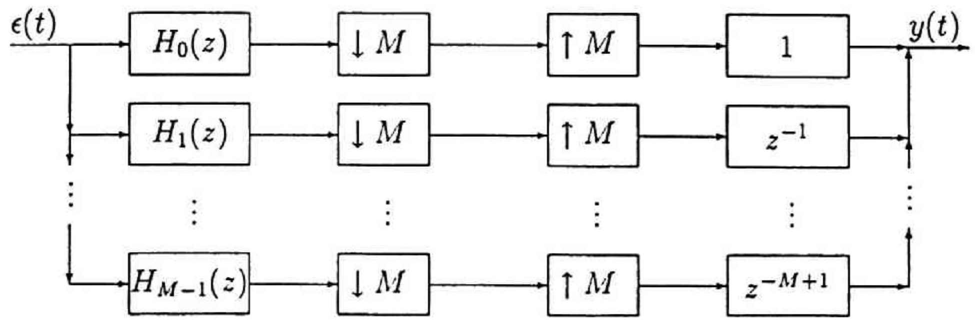

Figure 4.2: The LPTV system.

systems, (4.1) can be figured as Fig. 4.2. Therefore the LPTV system with period M can be interpreted as a special case of the M-band filter bank.

When each $H_k(z)$ is rational, that is,

$$H_t(z) = \frac{\mathcal{B}_t(z)}{\mathcal{A}_t(z)}, \tag{4.7}$$

we can write

$$\mathcal{A}_t(z) = \sum_{i=0}^{p_t} \alpha_t(i) z^{-i} \tag{4.8}$$

$$\mathcal{B}_t(z) = \sum_{i=0}^{q_t} \beta_t(i) z^{-i}. \tag{4.9}$$

where the coefficients are also periodic in time, that is,

$$p_t = p_{t+M}, q_t = q_{t+M} \tag{4.10}$$

$$\alpha_t(i) = \alpha_{t+M}(i) \quad \text{for } i = 0, \ldots, p_t \tag{4.11}$$

$$\beta_t(i) = \beta_{t+M}(i) \quad \text{for } i = 0, \ldots, q_t. \tag{4.12}$$

For convenience, we assume $\alpha_t(0) = 1$ and $b_t(0) = 1$ in the following.

For the stochastic input signal, we consider the following LPTV system

$$y(t) + \sum_{j=1}^{p_t} \alpha_t(j) y(t-j) = \epsilon(t) + \sum_{j=1}^{q_t} \beta_t(j) \epsilon(t-j) \tag{4.13}$$

where the coefficients satisfy (4.10), (4.11), and (4.12). In [35], $\epsilon(t)$ is assumed to be a cyclostationary white noise. Here we consider the more general case. The output $y(t)$ is called the periodic ARMA process with the orders

$$(p_0, \ldots, p_{M-1}; q_0, \ldots, q_{M-1}).$$

It is noted that the process whose $\beta_t(j) = 0$ in (4.13) and the process whose $\alpha_t(j) = 0$ in (4.13) are called the periodic AR process and the periodic MA process, respectively. These three processes are used to generate and to analyze the cyclostationary processes parametrically.

Assume $\epsilon(t)$ is cyclostationary with period M, zero mean and the spectral correlation density $F_\epsilon(\omega)$. Also for convenience, we put

$$\alpha_t(i) = 0 \quad i < 0 \text{ and } i > p_t \tag{4.14}$$

$$\beta_t(i) = 0 \quad i < 0 \text{ and } i > q_t. \tag{4.15}$$

Now we show the periodic ARMA process with period M can be transformed into the M-channel stationary process. By setting $t = k+nd$ (4.46) can be expressed as

$$y(k+nd) + \sum_{j=1}^{p_k} \alpha_k(j) y(k+nd-j) = \epsilon(k+nd) + \sum_{j=1}^{q_t} \beta_t(j)\epsilon(t-j) \tag{4.16}$$

for $k = 0, \ldots, M-1$. Construct two M-channel processes as (2.37) in Section 2.2 by

$$\begin{aligned} \boldsymbol{\epsilon}(t) &= (\epsilon(Mt), \epsilon(Mt+1), \ldots, \epsilon(Mt+M-1)^T & (4.17) \\ \boldsymbol{y}(t) &= (y(Mt), y(Mt+1), \ldots, y(Mt+M-1))^T & (4.18) \end{aligned}$$

which are shown in the left hand side and the right hand side in Fig. 4.3. It is noted that $\boldsymbol{\epsilon}(t)$ is M-channel stationary.

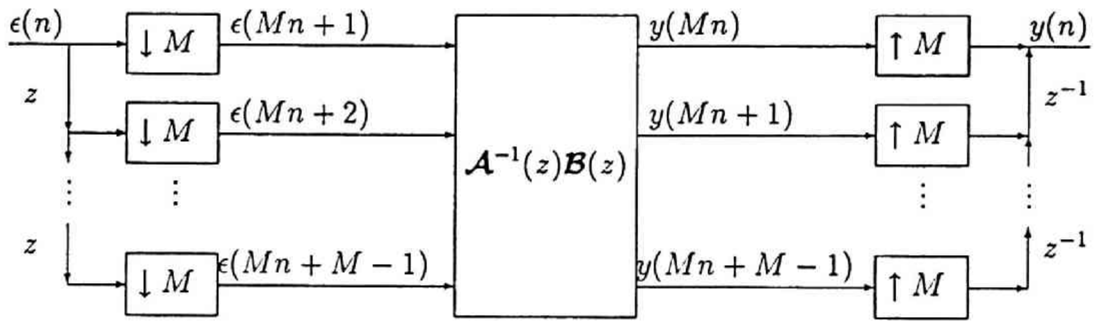

Figure 4.3: The input-output relation of the LPTV system

Define

$$\begin{aligned} p &= \max_i [\, \frac{(p_i - i)}{M} + 1 \,] & (4.19) \\ q &= \max_i [\, \frac{(q_i - i)}{M} + 1 \,] & (4.20) \end{aligned}$$

where, for integer j, $[\, x\,] = j$ for $j \le x < j+1$. Then (4.13) can be rewritten as

$$\mathcal{A}_0 \boldsymbol{y}(t) + \mathcal{A}_1 \boldsymbol{y}(t-1) + \cdots + \mathcal{A}_p \boldsymbol{y}(t-p) = \mathcal{B}_0 \boldsymbol{\epsilon}(t) + \mathcal{B}_1 \boldsymbol{\epsilon}(t-1) + \cdots + \mathcal{B}_q \boldsymbol{\epsilon}(t-q) \tag{4.21}$$

where

$$\mathcal{A}_0 = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ \alpha_1(1) & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & 0 \\ \alpha_{M-1}(M-1) & \alpha_{M-1}(M-2) & \cdots & 1 \end{pmatrix} \tag{4.22}$$

$$\mathcal{A}_j = \begin{pmatrix} \alpha_0(M(j-1)+M) & \cdots & \alpha_0(M(j-1)+1) \\ \alpha_1(M(j-1)+M+1) & \cdots & \alpha_1(M(j-1)+2) \\ \vdots & \ddots & \vdots \\ \alpha_{M-1}(M(j-1)+2M-1) & \cdots & \alpha_{M-1}(M(j-1)+M) \end{pmatrix} \quad j \neq 0 \tag{4.23}$$

and similarly for $\mathcal{B}_j$. Define

$$\begin{aligned} \boldsymbol{\mathcal{A}}(z) &= \sum_{n=0}^{p} \mathcal{A}_n z^{-n} && (4.24) \\ \boldsymbol{\mathcal{B}}(z) &= \sum_{n=0}^{q} \mathcal{B}_n z^{-n}. && (4.25) \end{aligned}$$

and Type 1 polyphase representation of $\mathcal{A}_k(z)$ and $\mathcal{B}_k(z)$ as

$$\begin{aligned} \mathcal{A}_k(z) &= \sum_{l=0}^{M-1} \mathcal{A}_{kl}(z^M) z^{-l} && (4.26) \\ \mathcal{B}_k(z) &= \sum_{l=0}^{M-1} \mathcal{B}_{kl}(z^M) z^{-l}. && (4.27) \end{aligned}$$

It can be easily shown that

$$\boldsymbol{\mathcal{A}}(z) = \begin{pmatrix} \mathcal{A}_{00}(z^M) & z^{-1}\mathcal{A}_{0,M-1}(z^M) & \cdots & z^{-1}\mathcal{A}_{01}(z^M) \\ \mathcal{A}_{11}(z^M) & \mathcal{A}_{10}(z^M) & \cdots & z^{-1}\mathcal{A}_{1,M-1}(z^M) \\ \vdots & \vdots & \cdots & \vdots \\ \mathcal{A}_{M-1,M-1}(z^M) & \mathcal{A}_{M-1,M-2}(z^M) & \cdots & \mathcal{A}_{M-1,0}(z^M) \end{pmatrix} \tag{4.28}$$

and similarly for $\boldsymbol{\mathcal{B}}(z)$.

If $\boldsymbol{\mathcal{A}}^{-1}(z)$ exists and is stable, from the the theory of multichannel stationary processes, $\boldsymbol{y}(t)$ is given by

$$\boldsymbol{y}(t) = \boldsymbol{\mathcal{A}}^{-1}(q)\boldsymbol{\mathcal{B}}(q)\boldsymbol{\epsilon}(t). \tag{4.29}$$

Since the input $\boldsymbol{\epsilon}(t)$ is M-channel stationary, so is the output $\boldsymbol{y}(n)$. Also since the M-channel process $\boldsymbol{y}(t)$ is stationary, the scalar process $y(t)$ is cyclostationary with period M. By using the notations of multirate systems, this system can be written as Fig. 4.3.

Premultiplying (4.21) by $\mathcal{A}_0^{-1}$, we have

$$y(t)+A_1y(t-1)+\cdots+A_py(t-p)=B_0\epsilon(t)+B_1\epsilon(t-1)+\cdots+B_q\epsilon(t-q) \quad (4.30)$$

where

$$A_i = \mathcal{A}_0^{-1}\mathcal{A}_i \quad \text{for } i=0,\ldots,p \quad (4.31)$$

$$B_i = \mathcal{B}_0^{-1}\mathcal{B}_i \quad \text{for } i=0,\ldots,q \quad (4.32)$$

This is an M input and M output multichannel ARMA(p,q) process. So it is shown that the periodic ARMA process can be transformed into the M channel ARMA process.

Next, we derive the spectral correlation density of the output of the periodic ARMA process (4.13). From Gladyshev's relation (3.48), $\epsilon(t)$ is an M-channel stationary process with the spectral density matrix,

$$S_\epsilon(\omega)=\frac{1}{M}\Lambda(e^{-j\frac{\omega}{M}})W^\dagger F_\epsilon(\frac{\omega}{M})W\Lambda(e^{j\frac{\omega}{M}}) \quad \text{for } |\omega|\leq\frac{\pi}{M}. \quad (4.33)$$

Since $\epsilon(t)$ is filtered by $\mathcal{A}^{-1}(z)\mathcal{B}(z)$ to produce $y(t)$, the spectral density matrix of $y(t)$ is given by

$$S_y(\omega)=\mathcal{A}^{-1}(z)\mathcal{B}(z)S_\epsilon(z)\left(\mathcal{A}^{-1}(z)\mathcal{B}(z)\right)^\dagger|_{z=e^{j\omega}}. \quad (4.34)$$

Also from Gladyshev's relation (3.48), the spectral correlation density matrix is given by

$$F_y(\omega) = \frac{1}{M}W\Lambda(e^{j\omega})S_\epsilon(M\omega)\Lambda(e^{-j\omega})W^\dagger \quad (4.35)$$

$$= \mathcal{P}(z)F_x(\omega)\mathcal{P}^\dagger(z)|_{z=e^{j\omega}} \quad (4.36)$$

where

$$\mathcal{P}(z)=\frac{1}{M}W\Lambda(z)\mathcal{A}^{-1}(z)\mathcal{B}(z)\Lambda(z^{-1})W^\dagger. \quad (4.37)$$

4.2 Periodic AR Processes

Periodic AR processes were originally studied by Pagano [31]. There it has been shown that there is a one-to-one correspondence between a multichannel AR process and a scalar periodic AR process. So periodic AR processes can be analyzed by the theory of multichannel AR processes, and vice versa.

Multichannel AR processes have been widely studied. And the LWR algorithm is well known for obtaining the predictor coefficient matrices of multichannel AR processes [47]. It has also been shown that backward multichannel AR processes can be constructed from the auxiliary matrices used in the LWR algorithm. By a similar correspondence, the coefficients of backward periodic AR processes can be also obtained by using the resulting multichannel backward AR processes. But, in this case, it is necessary to make two these transformations.

On the other hand, for directly obtaining the coefficients of periodic AR processes, the circular Levinson algorithm was derived by Sakai [33]. Several properties of periodic AR processes were studied by Sakai [34] [36] using this algorithm. But backward periodic AR processes equivalent to backward multichannel AR processes were not explicitly shown there.

In this section, we construct backward periodic AR processes from the auxiliary coefficients used in the circular Levinson algorithm. By virtue of this, we can obtain the coefficients of backward periodic AR processes without using the LWR algorithm. Furthermore, we show that the orders of the backward periodic AR process can be obtained from those of the corresponding forward one without the use of the circular Levinson algorithm. The statistical properties of the estimated coefficients of backward periodic AR processes based on a sample of finite size are also derived.

4.2.1 Multichannel AR processes

We give a brief review of multichannel AR processes. The d-channel process $x(t)$ is said to be a multichannel AR process with zero mean vector if

$$x(t) + \sum_{j=1}^{p} A(j)x(t-j) = s(t) \tag{4.38}$$

in which

$$E[\ s(t)\ ] = 0, \quad E[\ s(t)s^T(s)\ ] = \Sigma\delta_{t,s}. \tag{4.39}$$

where Σ is positive definite. We denote the auto-covariance matrices of $x(t)$ by

$$R(k) = E[\ x(t+k)x^T(t)\ ]. \tag{4.40}$$

With these covariance matrices, the coefficient matrices are given by solving the normal equations

$$[\ I \quad A(1) \quad \cdots \quad A(p)\ ]R^{(p)} = [\ \Sigma \quad 0 \quad \cdots \quad 0\ ]\,, \tag{4.41}$$

where $R^{(p)}$ is the $(p+1) \times (p+1)$ block matrix constructed by

$$\begin{pmatrix} R(0) & R(1) & \cdots & R(p) \\ R(-1) & R(0) & \cdots & R(p-1) \\ \vdots & \vdots & \ddots & \vdots \\ R(-p) & R(p-1) & \cdots & R(0) \end{pmatrix}. \tag{4.42}$$

The LWR algorithm is well known for successively solving the above equations [47].

The LWR algorithm

1. *Initial conditions*

$$A_0(0) = B_0(0) = I\ , \Sigma_0 = V_0 = R(0)$$

2. *Order update for* $j = 0, \ldots, p-1$

   (a) *compute*

$$\begin{aligned} \Delta_j &= \sum_{m=0}^{j} A_j(m)R(j+1-m) \\ A_{j+1}(j+1) &= -\Delta_j V_j^{-1} \\ B_{j+1}(j+1) &= -\Delta_j^T \Sigma_j^{-1} \end{aligned}$$

   (b) *update*

$$\Sigma_{j+1} = \Sigma_j + A_{j+1}(j+1)\Delta_j^T$$

$$
\begin{aligned}
V_{j+1} &= V_j + B_{j+1}(j+1)\Delta_j \\
\textit{for } i = 1, \ldots, j & \\
A_{j+1}(i) &= A_j(i) + A_{j+1}(j+1)B_j(j+1-i) \\
B_{j+1}(i) &= B_j(i) + B_{j+1}(j+1)A_j(j+1-i)
\end{aligned}
$$

3.

$$\Sigma = \Sigma_p, \quad V = V_p$$

$$A(i) = A_p(i), \; B(i) = B_p(i) \;\; (i = 1, \ldots, p)$$

It has been also shown that the auxiliary matrices used in this algorithm satisfy

$$[\; B(p) \;\; \cdots \;\; B(1) \;\; I \;]R^{(p)} = [\; 0 \;\; \cdots \;\; 0 \;\; V \;], \tag{4.43}$$

from which we can construct the backward multichannel AR process

$$x(t) + \sum_{j=1}^{p} B(j)x(t+j) = v(t) \tag{4.44}$$

where

$$E[\; v(t) \;] = 0, \quad E[\; v(t)v^T(s) \;] = V\delta_{t,s}. \tag{4.45}$$

4.2.2 The circular Levinson algorithm

Consider the following periodic AR process with period d and orders $(p_1, \ldots, p_d)$;

$$y(t) + \sum_{j=1}^{p_t} \alpha_t(j)y(t-j) = \epsilon(t) \tag{4.46}$$

where all coefficients satisfy (4.10) and (4.11) with M replaced by d. Let the input $\epsilon(t)$ be uncorrelated with zero mean and the variance at time t, denoted by $\sigma_\epsilon^2(t)$, which satisfies

$$\sigma_\epsilon^2(t) = \sigma_\epsilon^2(t+M). \tag{4.47}$$

We denote the covariance of $y(t)$ by

$$R(t,s) = E[\; y(t)y(s) \;]\,. \tag{4.48}$$

Since $y(t)$ is cyclostationary with period d, we have

$$R(t,s) = R(t+d, s+d). \tag{4.49}$$

Using (4.17), (4.18) and (4.21), (4.46) can be written by

$$\mathcal{A}_0 y(t) + \mathcal{A}_1 y(t-1) + \cdots + \mathcal{A}_p y(t-p) = \epsilon(t). \tag{4.50}$$

Then Pagano showed the following theorem [31].

Theorem 2

If $x(t)$ in (4.38) and $y(t)$ is associated by

$$x(t) = (y(td), y(1+td), \ldots, y(d-1+td))^T, \tag{4.51}$$

then $x(t)$ is a d-channel AR process of order p with positive definite Σ if and only if $y(t)$ is a periodic AR process of period d and orders $(p_1, \ldots, p_d)$ with positive $\sigma_0^2, \ldots, \sigma_{d-1}^2$ and $p = max_j[\,(p_j - j)/d + 1\,]$. And each coefficient is related by

$$\Sigma = LDL^T \tag{4.52}$$

$$L^{-1}A(j) = \mathcal{A}_j \tag{4.53}$$

where

$$D = diag(\sigma_0^2, \sigma_1^2, \ldots, \sigma_{d-1}^2) \tag{4.54}$$

$$L^{-1} = \mathcal{A}_0. \tag{4.55}$$

Proof

Premultiplying the periodic AR process (4.50) by $\mathcal{A}_0$, we obtain the d-channel AR process (4.38) whose coefficients satisfy (4.52) and (4.53).

Conversely, since Σ is positive definite, the unique LDL^T factorization can be obtained where L is unit lower triangular D is diagonal. By setting $\mathcal{A}_0 = L^{-1}$ and premultiplying (4.38) by L^{-1}, the periodic AR process can be obtained whose coefficients satisfy (4.52) and (4.53). □

This result enables us to analyze periodic AR processes by the theory of multichannel AR processes, and vice versa. Note that from (4.51) we have

$$(R(t-s))_{ik} = R(i+td, k+sd). \tag{4.56}$$

Similarly to (4.52), the unique $\boldsymbol{UDU}^T$ decomposition of $\boldsymbol{V}$ in (4.43) can be obtained where $\boldsymbol{U}$ is unit upper triangular and $\boldsymbol{D}$ is diagonal. By setting

$$\boldsymbol{U}^{-1} = \begin{pmatrix} 1 & \bar{\beta}_0(1) & \cdots & \bar{\beta}_0(d-1) \\ 0 & 1 & \cdots & \bar{\beta}_1(d-2) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 \end{pmatrix} \tag{4.57}$$

and premultiplying (4.44) by $\boldsymbol{U}^{-1}$, we have

$$\boldsymbol{U}^{-1}\boldsymbol{x}(t) + \sum_{j=1}^{p} \boldsymbol{\mathcal{B}}_j \boldsymbol{x}(t+j) = \bar{\eta}(t) \tag{4.58}$$

where

$$\bar{\eta}(t) = \boldsymbol{U}^{-1}\boldsymbol{v}(t)$$

$$\boldsymbol{\mathcal{B}}_j = \boldsymbol{U}^{-1}\boldsymbol{B}(j).$$

Now by setting

$$\bar{\eta}(t) = (\bar{\eta}(td), \bar{\eta}(1+td), \ldots, \bar{\eta}(d-1+td))^T \tag{4.59}$$

and

$$\begin{pmatrix} \bar{\beta}_0(d+(j-1)d) & \cdots & \bar{\beta}_0(2d-1+(j-1)d) \\ \bar{\beta}_1(d-1+(j-1)d) & \cdots & \bar{\beta}_1(2d-2+(j-1)d) \\ \vdots & \ddots & \vdots \\ \bar{\beta}_{d-1}(1+(j-1)d) & \cdots & \bar{\beta}_{d-1}(d+(j-1)d) \end{pmatrix} = \boldsymbol{\mathcal{B}}_j, \tag{4.60}$$

the kth row of (4.44) can be written by

$$y(k+nd) + \sum_{j=1}^{r_k} \bar{\beta}_k(j) y(k+nd+j) = \bar{\eta}(k+nd) \tag{4.61}$$

where $\bar{\eta}(k+nd)$ are uncorrelated with each other. Thus we can say that there exists the backward periodic AR process given by (4.61). Therefore, the coefficients of the (forward) periodic AR process and the backward one are given by obtaining coefficients of the corresponding multichannel AR process. But by this approach we have to make two transformations between them. Moreover we can not obtain the orders of the backward AR process in advance.

In order to directly obtain the coefficients of the periodic AR process, define the jth order kth channel forward and backward linear prediction errors for $y(t)$ by

$$\epsilon(j, k+nd) = y(k+nd) + \sum_{i=1}^{j} \alpha_k(j,i) y(k+nd-i) \tag{4.62}$$

and

$$\eta(j, k+nd) = y(k+nd-j) + \sum_{i=1}^{j} \beta_k(j, j+1-i) y(k+nd-i+1) \tag{4.63}$$

respectively. The predictor coefficients $\alpha_k(j,i)$, $\beta_k(j,i)$ for $i = 1, \ldots, j$ are determined by minimizing

$$E[\epsilon^2(j, k+nd)] := \sigma_k^2(j) \tag{4.64}$$

$$E[\eta^2(j, k+nd)] := \tau_k^2(j) \tag{4.65}$$

with respect to $\alpha_k(j,i)$ and $\beta_k(j,i)$, respectively. Then we have following normal equations

$$\boldsymbol{R}_k(j)\boldsymbol{\alpha}_k(j) = (\sigma_k^2(j), 0, \ldots, 0)^T \tag{4.66}$$

$$\boldsymbol{R}_k(j)\boldsymbol{\beta}_k(j) = (0, \ldots, 0, \tau_k^2(j))^T , \tag{4.67}$$

where

$$\begin{aligned}
\boldsymbol{\alpha}_k(j) &= (1, \alpha_k(j,1), \ldots, \alpha_k(j,j))^T, \\
\boldsymbol{\beta}_k(j) &= (\beta_k(j,j), \ldots, \beta_k(j,1), 1)^T, \\
(\boldsymbol{R}_k(j))_{p,q} &= R(k-p+1, k-q+1) \;\; (0 \le p, q \le j).
\end{aligned}$$

We assume $\boldsymbol{R}_k(p_k)$ is positive definite. Then, Sakai [33] derived the following efficient algorithm for successively obtaining $\boldsymbol{\alpha}_k(j)$, $\boldsymbol{\beta}_k(j)$ $(j = 1, 2, \ldots)$.

The Circular Levinson Algorithm

1. *Initial conditions ($j = 0$)*

$$\sigma_k^2(0) = \tau_k^2(0) = R(k,k), \quad \Delta_k(0) = R(k, k-1), \quad (k = 0, \ldots, d-1) \tag{4.68}$$

2. Order update from j to $j+1$

compute

$$\begin{aligned}\Delta_k(j) &= \sum_{m=0}^{j} R(k-m, k-j-1)\alpha_k(j,m) && (4.69)\\ \alpha_k(j+1,j+1) &= -\Delta_k(j)/\tau_{k-1}^2(j) && (4.70)\\ \beta_k(j+1,j+1) &= -\Delta_k(j)/\sigma_k^2(j) && (4.71)\end{aligned}$$

update

$$\sigma_k^2(j+1) = \sigma_k^2(j)\{1-\alpha_k(j+1,j+1)\beta_k(j+1,j+1)\} \quad (4.72)$$

$$\tau_k^2(j+1) = \tau_{k-1}^2(j)\{1-\alpha_k(j+1,j+1)\beta_k(j+1,j+1)\} \quad (4.73)$$

for $i = 1, \ldots, j$

$$\alpha_k(j+1,i) = \alpha_k(j,i) + \alpha_k(j+1,j+1)\beta_{k-1}(j,j+1-i) \quad (4.74)$$

$$\beta_k(j+1,i) = \beta_{k-1}(j,i) + \beta_k(j+1,j+1)\alpha_k(j,j+1-i) \quad (4.75)$$

where the subscript $k-1=-1$ is replaced by $d-1$.

If we define the normalized PARCOR coefficients by

$$\rho_k(j+1) = -\frac{\Delta_k(j)}{\sigma_k(j)\tau_{k-1}(j)} \quad (4.76)$$

then, from (4.72), we have $|\rho_k(j)| < 1$. Using (4.62), (4.63), (4.74) and (4.75), $\eta(j+1,k+nd)$ and $\epsilon(j+1,k+nd)$ can be expressed as

$$\begin{aligned}\epsilon(j+1,k+nd) &= \epsilon(j,k+nd) + \alpha_k(j+1,j+1)\eta(j,k-1+nd) && (4.77)\\ \eta(j+1,k+nd) &= \eta(j,k-1+nd) + \beta_k(j+1,j+1)\epsilon(j,k+nd)\,. && (4.78)\end{aligned}$$

Note that for a periodic AR process

$$\begin{aligned}\alpha(j+1,j+1) &= 0 && (4.79)\\ \sigma_k^2(j+1) &= \sigma_k^2 && (4.80)\end{aligned}$$

for all $j \geq p_k$. So from (4.77), we have

$$\epsilon(j+1,k+nd) = \epsilon(p_k,k+nd) \quad \text{for } j \geq p_k. \quad (4.81)$$

Also note that this $\epsilon(p_k, k+nd)$ is a white noise. From (4.70) and (4.71), we also have $\beta_k(j+1, j+1) = 0$ for all $j \geq p_k$ so that from (4.78)

$$\eta(j+1, k+nd) = \eta(p_k, k-1+nd) \quad \text{for } j \geq p_k. \tag{4.82}$$

But from this it is difficult to construct a backward white noise.

4.2.3 Backward periodic AR processes

In order to construct a backward periodic AR process, we want to find the backward periodic linear prediction error filter whose errors are white noises. We predict $y(k+nd)$ from $y(k+nd+1), \ldots, y(k+nd+j)$ and define new linear prediction error as

$$\bar{\eta}(j, k+nd) = y(k+nd) + \sum_{i=1}^{j} \bar{\beta}_k(j,i) y(k+nd+i) \ . \tag{4.83}$$

Minimizing

$$E[\, \bar{\eta}^2(j, k+nd) \,] = \bar{\tau}_k^2(j) \tag{4.84}$$

with respect to $\bar{\beta}_k(j,i)$, we have the following normal equations

$$\boldsymbol{R}_{k+j}(j)\bar{\beta}_k(j) = (0, \ldots, 0, \bar{\tau}_k^2(j))^T \tag{4.85}$$

where

$$\bar{\beta}_k(j) = (\bar{\beta}_k(j,j), \ldots, \bar{\beta}_k(j,1), 1)^T. \tag{4.86}$$

From (4.67), we also have

$$\boldsymbol{R}_{k+j}(j)\beta_{k+j}(j) = (0, \ldots, 0, \tau_{k+j}^2(j))^T \tag{4.87}$$

Also from positive definiteness of $\boldsymbol{R}_{k+j}(j)$ and replacing k in (4.67) with $k+j$, we have

$$\bar{\beta}_k(j) = \beta_{k+j}(j) \tag{4.88}$$

$$\bar{\tau}_k^2(j) = \tau_{k+j}^2(j). \tag{4.89}$$

Then from (4.78), (4.83) and (4.88),

$$\begin{aligned}
\bar{\eta}(j, k+nd) &= y(k+nd) + \sum_{i=1}^{j} \bar{\beta}_k(j,i) y(k+nd+i) && (4.90)\\
&= y(k+nd) + \sum_{i=1}^{j} \beta_{k+j}(j,i) y(k+nd+i) && (4.91)\\
&= \eta(j, k+j+nd). && (4.92)
\end{aligned}$$

Therefore, from (4.78) and (4.90) we get

$$\begin{aligned}\bar{\eta}(j+1,k+nd) &= \eta(j+1,k+j+1+nd) \\ &= \eta(j,k+j+nd)+\beta_{k+j}(j+1,j+1)\epsilon(j,k+j+1+nd) \\ &= \bar{\eta}(j,k+nd)+\bar{\beta}_k(j+1,j+1)\epsilon(j,k+j+1+nd)\,. \quad (4.93)\end{aligned}$$

It follows that, if we can find r_k that satisfys $\bar{\beta}_k(j+1,j+1)=0$ for all $j\geq r_k$, then

$$\bar{\eta}(j,k+nd)=\bar{\eta}(r_k,k+nd):=\bar{\eta}(k+nd).$$

Moreover, from

$$E[\,\bar{\eta}(k+nd)y(k+nd+i)\,]=0$$

for any positive integer i, we have

$$E[\bar{\eta}(k_1+n_1d)\bar{\eta}(k_2+n_2d)]=\delta_{k_1k_2}\delta_{n_1n_2}\bar{\tau}^2_{k_1}(r_{k_1}) \quad (4.94)$$

Thus we obtain a backward periodic AR process given by

$$y(k+nd)+\sum_{j=1}^{r_k}\bar{\beta}_k(j)y(k+nd+j)=\bar{\eta}(k+nd) \quad (4.95)$$

where

$$\bar{\beta}_k(j)=\bar{\beta}_{k+d}(j)=\beta_{k+r_k}(r_k,j)$$

for $j=1,\ldots,r_k$, each $\bar{\eta}(t)$ is uncorrelated with zero mean and variance $\bar{\tau}^2_k(r_k)=\tau^2_{k+r_k}(r_k)$.

Now we show how to obtain these r_k. At first, for fixed k, we classify the sequence $\bar{\beta}_k(j,j)$ into the d subsequences according to the remainder of j divided by d. By setting $j=i+ld$ $(i=0,1,\ldots,d-1,\ l=0,1,\ldots)$, from

$$\bar{\beta}_k(j,j)=\beta_{k+j}(j,j)=\beta_{k+i}(i+ld,i+ld)\,,$$

we know that the ith subsequence is constructed by $\beta_{k+i}(i+ld,i+ld)$.

Noting that

$$\beta_{k+i}(j+1,j+1)=0 \quad \text{for } j\geq p_{k+i}$$

for the ith subsequence, we have an l_i such that

$$\beta_{k+i}(i+ld,i+ld)\neq 0 \ \text{ and } \ \beta_{k+i}(i+(l+1)d,i+(l+1)d)=0 \ \text{ for all } \ l_i\leq l \quad (4.96)$$

that is,

$$i + l_i d \leq p_{k+i} < i + (l_i + 1)d.$$

It can be easily shown that this l_i is given by

$$l_i = \lfloor \frac{p_{k+i} - i}{d} \rfloor . \tag{4.97}$$

Thus if we put

$$r_k = \max_{0 \leq i \leq d-1} (i + l_i d), \tag{4.98}$$

then from (4.96) we have for a certain ith subsequence

$$\beta_{k+i}(r_k, r_k) \neq 0$$

and for all subsequences

$$\beta_{k+i}(j, j) = 0, \ \ j > r_k \ .$$

It follows that $\bar{\beta}_k(j+1, j+1) = 0$ for all $j \geq r_k$. Therefore we have the following algorithm to compute the orders of the backward periodic AR process from those of the (forward) periodic AR process.

for $k = 0, \ldots, d-1$

1. *Each* $i = 0, \ldots, d-1$, *let* $l_i = \lfloor (p_{k+i} - i) / d \rfloor$.

2. *For these* l_i*'s, let* $r_k = \max_{0 \leq i \leq d-1} (i + l_i d)$.

It should be noted that the above algorithm only needs the orders of periodic AR processes. So, we can obtain the orders of the backward one without calculating the coefficients. Also note that the orders of the backward one is not necessarily the same orders of the forward one.

4.2.4 An example

To illustratate the above results, a numerical example is presented. Sakai [34] has shown that there is a one-to-one correspondence between

$$R(k, k-j), (k = 0, \ldots, d-1; j = 0, 1, \ldots, p_k)$$

and

$$R(k,k), \rho_k(j), (k=0,\dots,d-1; j=0,1,\dots,p_k)$$

satisfying the condition $R(k,k)>0$ and $|\rho_k(j+1)|<1$. Then, giving the values

$$\begin{aligned}
&(R(0,0), R(1,1), R(2,2)) = (4,4,4),\\
&(\rho_0(1), \rho_0(2), \rho_0(3)) = (0.9, 0.6, 0.3),\\
&(\rho_1(1), \dots, \rho_1(5)) = (0.9, 0.8, 0.7, 0.6, 0.5),\\
&(\rho_2(1), \dots, \rho_2(4)) = (0.8, 0.6, 0.4, 0.2)\ ,
\end{aligned}$$

we obtain $R(k,k-j)$ $(j=0,1,\dots)$. In this case, the orders of the periodic AR process is

$$(p_0, p_1, p_2) = (3,5,4).$$

From our algorithm, those of the backward periodic AR process are given by

$$(r_0, r_1, r_2) = (4,4,5).$$

To certify our results on backward processes, with these $R(k,k-j)$, we calculate the coefficients by the circular Levinson algorithm and by the LWR algorithm, respectively.

By solving the normal equations by the LWR algorithm, the (forward) multichannel AR process is given by

$$\begin{aligned}
x(t) \ + & \begin{pmatrix} 0.300 & 0.811 & 1.379 \\ 0.287 & -0.450 & -1.079 \\ -1.535 & -0.874 & 0.240 \end{pmatrix} x(t-1) \\
+ & \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0.312 \\ 0 & 0 & -0.767 \end{pmatrix} x(t-2) = s(t)
\end{aligned} \tag{4.99}$$

where

$$\Sigma = \begin{pmatrix} 0.443 & 0.901 & 0.995 \\ -0.901 & 1.902 & -2.191 \\ 0.995 & -2.191 & 3.386 \end{pmatrix} = LDL^T$$

$$\boldsymbol{L} = \begin{pmatrix} 1 & 0 & 0 \\ -2.036 & 1 & 0 \\ 2.247 & -2.460 & 1 \end{pmatrix}, \boldsymbol{D} = \begin{pmatrix} 0.442 & 0 & 0 \\ 0 & 0.067 & 0 \\ 0 & 0 & 0.743 \end{pmatrix}.$$

Premultiplying both sides of (4.99) by $\boldsymbol{L}^{-1}$, we obtain

$$\begin{pmatrix} 1 & 0 & 0 \\ 2.036 & 1 & 0 \\ 2.760 & 2.460 & 1 \end{pmatrix} \boldsymbol{x}(t) + \begin{pmatrix} 0.300 & 0.811 & 1.379 \\ 0.898 & 1.200 & 1.730 \\ 0 & 0.257 & 1.394 \end{pmatrix} \boldsymbol{x}(t-1)$$

$$+ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0.312 \\ 0 & 0 & 0 \end{pmatrix} \boldsymbol{x}(t-2) = \begin{pmatrix} \epsilon(3t) \\ \epsilon(1+3t) \\ \epsilon(2+3t) \end{pmatrix}.$$

And the backward multichannel AR process is given by

$$\boldsymbol{x}(t) + \begin{pmatrix} 1.797 & 2.726 & 3.606 \\ -2.338 & -3.258 & -4.009 \\ 1.797 & 2.726 & 3.606 \end{pmatrix} \boldsymbol{x}(t+1)$$

$$+ \begin{pmatrix} 4.091 & 2.009 & 0.000 \\ -4.386 & -2.154 & 0.000 \\ 1.633 & 0.802 & 0.000 \end{pmatrix} \boldsymbol{x}(t+2) = \boldsymbol{v}(t)$$

where

$$\boldsymbol{V} = \begin{pmatrix} 2.806 & -1.966 & 0.432 \\ -1.966 & 1.694 & -0.463 \\ 0.432 & -0.463 & 0.172 \end{pmatrix} = \boldsymbol{U}\boldsymbol{D}\boldsymbol{U}^T$$

$$\boldsymbol{U} = \begin{pmatrix} 1 & -1.787 & 2.505 \\ 0 & 1 & -2.865 \\ 0 & 0 & 1 \end{pmatrix}, \boldsymbol{D} = \begin{pmatrix} 0.172 & 0 & 0 \\ 0 & 0.451 & 0 \\ 0 & 0 & 0.283 \end{pmatrix}$$

Premultiplying both sides of (4.100) by U^{-1}, we obtain

$$\begin{pmatrix} 1 & 1.787 & 2.294 \\ 0 & 1 & 2.686 \\ 0 & 0 & 1 \end{pmatrix} \boldsymbol{x}(t) + \begin{pmatrix} 2.368 & 1.069 & 0 \\ 3.221 & 1.618 & 0.156 \\ 2.070 & 1.815 & 1.551 \end{pmatrix} \boldsymbol{x}(t+1)$$

$$+\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1.633 & 0.802 & 0 \end{pmatrix} x(t+2) = \begin{pmatrix} \bar{\eta}(3t) \\ \bar{\eta}(1+3t) \\ \bar{\eta}(2+3t) \end{pmatrix}.$$

The results by using circular Levinson algorithm are identical with the above results, as it should be.

4.2.5 Statistical properties

Finally, we present the statistical properties of the estimates of $\bar{\beta}_k(r_k)$. Given $y(1)$, ..., $y(Nd)$ from a zero mean Gaussian periodic AR process, we use the estimate of $R(k,v)$ given by

$$R_N(k,v) = N^{-1}\sum_{j=0}^{m} y(k+jd)y(v+jd) \tag{4.100}$$

where $m = [N - max(k,v)/d]$. The symbol with a hat denotes the estimate of its corresponding quantity. It was shown by Pagano [31] that, as $N \to \infty$, $N^{\frac{1}{2}}\{R_N(k,v) - R(k,v)\}$ are asymptotically Gaussian with zero mean and covariance

$$N \operatorname{Cov}\{R_N(k_1,v_1), R_N(k_2,v_2)\} =$$
$$\sum_{u=-\infty}^{\infty} \{R(k_1,k_2+ud)R(v_1,v_2+ud) + R(k_1,v_2+ud)R(v_1,k_2+ud)\} \tag{4.101}$$

and $N^{\frac{1}{2}}(\widehat{\alpha}_k(1) - \alpha_k(1), \ldots, \widehat{\alpha}_k(p_k) - \alpha_k(p_k))^T$ for $k = 0, \ldots, d-1$ are asymptotically Gaussian with zero mean and covariance $\sigma_k^2 R_{k-1}^{-1}(p_k - 1)$, and are uncorrelated with each channel.

Similarly the statistical properties of the estimates of $\widehat{\bar{\beta}}_k(r_k)$ can be derived as follows: The normal equations (4.85) are written by

$$R(k,k+v) + \sum_{j=1}^{r_k} \bar{\beta}_k(j)R(k+j,k+v) = \delta_{v0}\bar{\tau}_k^2(r_k) \quad (v \geq 0). \tag{4.102}$$

With $R_N(k,v)$, the estimates of coefficients are given as the solution of the normal equation

$$R_N(k,k+v) + \sum_{j=1}^{r_k} \widehat{\bar{\beta}}_k(j)R_N(k+j,k+v) = \delta_{v0}\widehat{\bar{\tau}}_k^2(r_k) \quad (v \geq 0). \tag{4.103}$$

From (4.102) and (4.103), for $v = 1, \ldots, r_k$; $k = 0, \ldots, d-1$ we have

$$\sum_{j=1}^{r_k} R_N(k+j, k+v)\{\hat{\bar{\beta}}_k(j) - \bar{\beta}_k(j)\} = -\sum_{j=0}^{r_k}\{R_N(k+j, k+v) - R(k+j, k+v)\}\bar{\beta}_k(j). \tag{4.104}$$

From (4.101), the covariance of the right hand side of the above equation are given by

$$\begin{aligned}
&\sum_{j_1=0}^{r_{k_1}}\sum_{j_2=0}^{r_{k_2}} \bar{\beta}_{k_1}(j_1)\bar{\beta}_{k_2}(j_2)\mathrm{Cov}\{R_N(k_1+j_1, k_1+v_1), R_N(k_2+j_2, k_2+v_2)\} = \\
&N^{-1}\sum_{j_1,j_2}\bar{\beta}_{k_1}(j_1)\bar{\beta}_{k_2}(j_2)\sum_u\{R(k_1+j_1, k_2+j_2+ud)R(k_1+v_1, k_2+v_2+ud) \\
&+R(k_1+j_1, k_2+v_2+ud)R(k_1+v_1, k_2+j_2+ud)\}
\end{aligned} \tag{4.105}$$

for $v_1, v_2 = 1, \ldots, r_k$.

By noting (4.94) and (4.95) and by exchanging the order of summation and expectation, the first term of the right hand side of (4.105) is given by

$$\begin{aligned}
&N^{-1}\sum_{j_1,j_2,u}\bar{\beta}_{k_1}(j_1)\bar{\beta}_{k_2}(j_2)R(k_1+j_1, k_2+j_2+ud)R(k_1+v_1, k_2+v_2+ud) \\
&= N^{-1}\sum_u E[\eta(k_1)\eta(k_2+ud)]R(k_1+v_1, k_2+v_2+ud) \\
&= N^{-1}\bar{\tau}_k^2\delta_{k_1k_2}R(k_1+v_1, k_2+v_2)
\end{aligned} \tag{4.106}$$

and the second term by

$$\begin{aligned}
&N^{-1}\sum_{j_1,j_2,u}\bar{\beta}_{k_1}(j_1)\bar{\beta}_{k_2}(j_2)R(k_1+j_1, k_2+v_2+ud)R(k_1+v_1, k_2+j_2+ud) \\
&= N^{-1}\sum_u E[y(k_2+v_2)\eta(k_1)]E[y(k_1+v_1)\eta(k_2+ud)] = 0
\end{aligned}$$

because this is nonzero if and only if $k_2 - k_1 + v_2 < 0$, $k_1 - k_2 + v_1 < 0$ but this is impossible since $v_1, v_2 > 0$.

The left side of (4.104) can be approximated as

$$\sum_{j=1}^{r_k} R_N(k+j, k+v)\{\hat{\bar{\beta}}_k(j) - \bar{\beta}_k(j)\} \approx \sum_{j=1}^{r_k} R(k+j, k+v)\{\hat{\bar{\beta}}_k(j) - \bar{\beta}_k(j)\}\ . \tag{4.107}$$

Therefore from (4.106) and (4.107) we can show that $N^{\frac{1}{2}}(\hat{\bar{\beta}}_k(1) - \bar{\beta}_k(1), \ldots, \hat{\bar{\beta}}_k(r_k) - \bar{\beta}_k(r_k))^T$ for $k = 0, \ldots, d-1$ are asymptotically Gaussian with zero mean and covariance $\bar{\tau}_{r_k}^2 \boldsymbol{R}_{k+r_k}^{-1}(r_k - 1)$, and are uncorrelated with each channel.

Chapter 5

CONCLUSION

In this thesis, the cyclostationary processes among non-stationary processes are studied and their applications to time delay estimation, optimization of filter banks and backward periodic AR process are provided. The results presented in this thesis are summarized below.

In Chapter 2, the sampling theorem of cyclostationary processes are explicitly presented. This is considered to be useful to analyze continuous-time cyclostationary processes by digital computers. A new derivation of Gladyshev's relation is also presented. This shows the spectral relation between discrete-time cyclostationary processes and discrete-time multichannel stationary processes. As one direct application of cyclostationary processes, the maximum likelihood estimator is derived by using the sampling theorem and the cyclostationary analysis for a problem of estimating the time difference of arrival of a communication signal with additive noises between two sensors. The performance of this method is better than the existing ones under some conditions. This is clarified by the results of the computer simulations.

In Chapter 3, since the output of a filter bank for a stationary input is cyclostationary, the general relation between the spectral density of the input and the spectral correlation density of the output is derived by using the cyclostationary spectral analysis. From this analysis, it is shown that that the output of an alias free filter bank for any stationary input is stationary. Then the perfect reconstruction (PR) condition is restated from the stochastic point of view. Also using the cyclostationary spectral analysis, a criterion is derived to optimize a two-band filter bank under the PR condition that minimizes the averaged variance of the reconstruction error when the high pass band signal is dropped. The criterion is easily computed by the covariance of the input signal and filter coefficients. From this criterion, the optimal biorthogonal filter banks are obtained. By adding additional constraints to the filter coefficients, the criterion of CQF banks that are orthogonal and that of PR LPF banks that have linear phase are respectively obtained. The obtained PR filter banks are compared in terms of other criteria.

In Chapter 4, the backward periodic AR processes are constructed from the auxiliary coefficients used in the circular Levinson algorithm. The orders of the backward periodic AR process are shown to be different from those of the corresponding periodic AR process. A numerical example and statistical property of the estimated coefficients from a sample of finite size are also presented.

By using the results derived in this thesis, further applications and analysis of cyclostationary processes will be made. For time delay estimation, it is expected that better estimates will be obtained if the sensor array with multiple sensors is used. Moreover the attempt of estimating the two-dimensional time delay of a cyclostationary signal should be made. For filter banks in multirate systems, the stochastic analysis for one-dimensional signals is studied in this thesis. But it is also necessary to study that for two-dimensional signals since they are used in video signal processing. Although a question remains whether the results in this thesis can be directly extended to the two-dimensional signals, many useful applications will be generated. For backward periodic AR processes, the analysis of actual data should be made by using the results in this thesis. And further research on parametric analysis for cyclostationary processes has to be continued.

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